Note on the transformation that reduces the Burgers equation to the heat equation.

We started to be interested in the question of who first discovered the reduction of the Burgers equation to the heat equation (known as the Hopf-Cole transformation) since in various recent publications it was pointed out that Hopf and Cole were not the first to discover it. In particular it was claimed that this transformation appeared in the work of Forsyth at the beginning of the XX century and in the work of Florin in the forties. While the reference to Florin is correct, that to Forsyth unfortunately is not. In the work of Forsyth some formulas similar to those occurring in the Hopf-Cole transformation appear (and this has perhaps caused some confusion), but the transformation itself does not. However, the theorem obtained by Forsyth is itself very interesting, and we review it in §1. In §2 we explain the transform in the multidimensional case and in §3 we give a review of Florin's work. It is interesting to note that this transformation reduces the question of uniqueness for the Burgers equation to the Widder uniqueness theorem for the heat equation in the class of positive solutions. Since we have failed to find a proof of Widder's uniqueness theorem for the multidimensional case in the literature, we include one in the Appendix.

§1. Review of subsections 206 and 207 of the Forsyth's book [1]

Consider a *linear* second order equation for an unknown function $u = u(t,x)$:

$$\nu(t,x)\tfrac{\partial^2 u}{\partial x^2} - \alpha(t,x)\tfrac{\partial u}{\partial x} - \tfrac{\partial u}{\partial t} + \gamma(t,x)u = 0. \tag{1}$$

By the *linear* substitution

$$u = \lambda(t,x)v, \tag{2}$$

where $\lambda = \lambda(t,x)$ is a fixed (known) function and $v = v(t,x)$ is a new unknown function we transform the equation (1) to the new, again *linear* equation

$$\tilde\nu(t,x)\tfrac{\partial^2 v}{\partial x^2} - \tilde\alpha(t,x)\tfrac{\partial v}{\partial x} - \tfrac{\partial v}{\partial t} + \tilde\gamma(t,x)v = 0,$$

where

$$\begin{aligned}
&\tilde\nu = \nu\,,\\
&\tilde\alpha = \alpha - \tfrac{2\nu\lambda_x}{\lambda},\\
&\tilde\gamma = \gamma - \tfrac{\alpha\lambda_x}{\lambda} - \tfrac{\lambda_t}{\lambda} + \tfrac{\nu\lambda_{xx}}{\lambda}.
\end{aligned}$$

Hence the function $I(t,x) = \nu(t,x)$ is obviously an invariant under the transformations of the type (2). Forsyth found one more:

$$J(t,x) = \tfrac{\partial}{\partial x}\left(2\gamma + \nu\left(\tfrac{\alpha}{\nu}\right)_x - \tfrac12\tfrac{\alpha^2}{\nu}\right) - \tfrac{\partial}{\partial t}\left(\tfrac{\alpha}{\nu}\right).$$

Indeed, we have:

$$2\tilde\gamma + \tilde\nu\left(\tfrac{\tilde\alpha}{\tilde\nu}\right)_x - \tfrac12\tfrac{(\tilde\alpha)^2}{\tilde\nu} = 2\gamma + \nu\left(\tfrac{\alpha}{\nu}\right)_x - \tfrac12\tfrac{\alpha^2}{\nu} - \tfrac{2\lambda_t}{\lambda},$$

whence,

$$\tfrac{\partial}{\partial x}\left(2\tilde\gamma + \tilde\nu\left(\tfrac{\tilde\alpha}{\tilde\nu}\right)_x - \tfrac12\tfrac{(\tilde\alpha)^2}{\tilde\nu}\right) - \tfrac{\partial}{\partial t}\left(\tfrac{\tilde\alpha}{\tilde\nu}\right) = \tfrac{\partial}{\partial x}\left(2\gamma + \nu\left(\tfrac{\alpha}{\nu}\right)_x - \tfrac12\tfrac{\alpha^2}{\nu}\right) - \tfrac{\partial}{\partial t}\left(\tfrac{\alpha}{\nu}\right).$$

These invariants are found in subsection 206. In the subsection 207 Forsyth shows that I and J are the full system of invariants for equations of type (1) w.r.t. transformations (2). In other words, if we are given two equations of the type (1) such that the functions I and J for the first equations coincide with the corresponding functions for the second equation, then these equations can be transformed one into another by a substitution (2).

We now formulate a particular case of this fact.

Theorem. *An equation* (1) *can be reduced to the form*

$$v_t = \nu(t,x)v_{xx}$$

by the substitution (2) *iff* $J(t,x) \equiv 0$. *In the last case we have* $\lambda(t,x) = e^{\theta(t,x)}$, *where* $\theta(t,x)$ *is a function such that*

$$\frac{\partial\theta}{\partial x} = \frac{\alpha}{2\nu},$$
$$\frac{\partial\theta}{\partial t} = \gamma + \frac{\nu}{2}\left(\frac{\alpha}{\nu}\right)_x - \frac{1}{4}\frac{\alpha^2}{\nu}.$$

If $\nu = Const$, then the condition $J(t,x) = 0$ is nothing but the Burgers equation for α:

$$\alpha_t + \alpha\alpha_x = \nu\alpha_{xx} + 2\nu\gamma_x\,.$$

We note (see. §2), that by the substitution $\alpha = -2\nu\left(\ln\varphi\right)_x$ (i.e. $\varphi = \lambda^{-1}$) it can be reduced to the equation: $\varphi_t - \nu\Delta\varphi + \gamma\varphi = 0$.

Remark 1. Forsyth used some different notations. In particular, he normalises the coefficient in front of u_{xx} to be one, while we do this for the coefficient of u_t. Consequently the form of invariants I and J is changed.

Remark 2. The Burgers Equation, as a secondary condition for fulfilling certain conditions under some linear transformations, appears in [1] also, e.g., in exercise 3 on the page 102.

Although the Burgers equation was known to Forsyth, he did not investigated the possibility of reducing the non-linear Burgers equation to the liner one.

So far the first (known) appearance in the mathematical literature of such a transformation is in the paper by Florin [2]. In the next section we explain this transform and in the third section we give a review of what Florin actually did.

§2. The substitution. The multidimensional Burgers equation can be written in the two different forms:

$$\frac{\partial \boldsymbol{v}}{\partial t} + \nabla\frac{(\boldsymbol{v},\boldsymbol{v})}{2} = \nu\Delta\boldsymbol{v} \qquad \text{or} \qquad \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v},\nabla)\boldsymbol{v} = \nu\Delta\boldsymbol{v}\,. \tag{3}$$

In general $\nabla\frac{(\boldsymbol{v},\boldsymbol{v})}{2} \neq (\boldsymbol{v},\nabla)\boldsymbol{v}$. However we consider the potential case only for which these terms coincide. Here $\boldsymbol{v} = \boldsymbol{v}(t,\boldsymbol{x}) \in \mathbb{R}^n$, $t \geqslant 0$, $\boldsymbol{x} \in \mathbb{R}^n$. We recall that $(\boldsymbol{v},\nabla)$ is the operator of differentiating along the vector field $\boldsymbol{v}$.

Since we are considering the potential case, the initial conditions are required to be potential:

$$\boldsymbol{v}(0,\boldsymbol{x}) = \boldsymbol{v}_0(\boldsymbol{x}) = \nabla H_0(\boldsymbol{x})\,. \tag{4}$$

We assume that the function $H_0 : \mathbb{R}^n \to \mathbb{R}$ is continuous and that $\liminf\limits_{|\boldsymbol{x}|\to\infty} \frac{H_0(\boldsymbol{x})}{|\boldsymbol{x}|^2} \geqslant 0$.

The last condition is required to insure the existence of a solution for all $t \geqslant 0$. If, e.g., $\liminf\limits_{|\boldsymbol{x}|\to\infty} \frac{H_0(\boldsymbol{x})}{|\boldsymbol{x}|^2} \geqslant -k$, then we can insure existence only for $2kt < 1$. Example: Let $\boldsymbol{v}_0(\boldsymbol{x}) = -2k\boldsymbol{x}$, then $\boldsymbol{v}(t,\boldsymbol{x}) = \frac{-2k\boldsymbol{x}}{1-2kt}$ and the solution fails to exist for $2kt \geqslant 1$. The necessary and sufficient condition for global existence of the solution is the convergence of the integral (6) for each $t > 0$ and $\boldsymbol{x} \in \mathbb{R}^n$.

We define $\varphi_0(\boldsymbol{x}) = \exp\left(\frac{H_0(\boldsymbol{x})}{-2\nu}\right)$. Let $\varphi(t,\boldsymbol{x})$ is the solution of the Cauchy problem for the heat equation:

$$\varphi_t = \nu\Delta\varphi\,, \qquad \varphi(0,\boldsymbol{x}) = \varphi_0(\boldsymbol{x})\,, \tag{5}$$

i.e.

$$\varphi(t,\boldsymbol{x}) = \frac{1}{(4\pi t\nu)^{n/2}}\int_{\mathbb{R}^n}\varphi_0(z)\exp\left(\frac{-(x-z)^2}{4t\nu}\right)d\boldsymbol{z}\,. \tag{6}$$

We note that (5) has no more than one solution in the class of positive functions. In [5, chpt 8, §2] this is proven for the 1D case. However the proof can be easily extended to the multidimensional case without essential changes (see Appendix).

As a corollary we get that for *any* initial conditions the Burgers equation admits no more than one solution.

Theorem. *With our notations, the function* $\boldsymbol{v} = -2\nu\nabla\ln\varphi = -2\nu\frac{\nabla\varphi}{\varphi}$ *is a solution of the Cauchy problem (3), (4).*

Proof. Indeed:

$$\frac{\partial \boldsymbol{v}}{\partial t} + \nabla\frac{(\boldsymbol{v},\boldsymbol{v})}{2} - \nu\Delta\boldsymbol{v} = -2\nu\nabla\left(\frac{\varphi_t - \nu\Delta\varphi}{\varphi}\right). \qquad \square$$

The uniqueness result for Burgers equation follows from the Widder's uniqueness theorem and the observation:

Let a potential vector field $\boldsymbol{v}$ *satisfies the (3). Take any potential* H_0 *of the initial state, i.e.,* $\boldsymbol{u}(0,\cdot) = \nabla H_0$. *Then there exists a positive function* φ *such that*

$$\varphi_t = \nu\Delta\varphi\,, \qquad \varphi(0,\boldsymbol{x}) = \exp\left(\frac{H_0(\boldsymbol{x})}{-2\nu}\right) \quad \textit{and} \quad \boldsymbol{v} = -2\nu\nabla\ln\varphi\,.$$

It is interesting to note that for the uniqueness theorem for the non-linear Burgers equation no assumptions on growth of the initial conditions should be made. Whereas for the linear heat equation we should put some restrictions, e.g., the Tikhonov growth restrictions.

§3 Florin's work. If a potential vector field satisfies (3) then there exists a potential H (i.e. $\boldsymbol{v} = \operatorname{grad} H$) that satisfies the following equation:

$$\frac{\partial H}{\partial t} + \frac{1}{2}(\operatorname{grad} H, \operatorname{grad} H) = \nu\Delta H.$$

In the late forties V. A. Florin, considering the problem of consolidation of wet soil, arrived at the equation (the 3D case):

$$\frac{\partial H}{\partial t} + \alpha(\operatorname{grad} H)^2 + \beta(\operatorname{grad} H, \operatorname{grad}\psi) + \delta\nabla^2 H + \frac{\partial F}{\partial t} = 0, \qquad [2,\ \text{eq. } (17)]$$

where H is the hydrostatic pressure, $\alpha, \beta, \delta, \psi$, and $\partial F/\partial t$ are certain functions of position and time. In [2] on the page 1392, under the assumption $\alpha/\delta = const$, Florin makes the substitution

$$H = \frac{\delta}{\alpha}\ln(\varphi + C) + D, \qquad [2,\ \text{eq. } (18)]$$

where C and D are some constants, and reduces the equation above to the linear equation:

$$\frac{\partial\varphi}{\partial t} + \beta(\operatorname{grad}\varphi, \operatorname{grad}\psi) + \delta\nabla^2\varphi + \frac{\alpha}{\delta}(\varphi + C)\frac{\partial F}{\partial t} = 0. \qquad [2,\ \text{eq. } (19)]$$

Later, this transformation[1] was independently rediscovered by E. Hopf [3] and J. Cole [4].

Appendix. The uniqueness theorem for the heat equation in the class of non-negative functions.

For the 1D case the uniqueness theorem was proved by Widder [5], [6]. We reproduce his proof for the multidimensional case.

Theorem. *Suppose that a non-negative continuous function* $u = u(t,\boldsymbol{x})$ *is defined on the strip* $[0,T)\times\mathbb{R}^n$, *and suppose that for* $t > 0$ *this function is* $C^{1,2}$*-smooth and satisfies the equation* $u_t = \Delta u$. *Then for any* $t \in (0,T)$ *and* $\boldsymbol{x} \in \mathbb{R}^n$ *we have:*

$$u(t,\boldsymbol{x}) = \int_{\mathbb{R}^n} G(t,\boldsymbol{x}-\boldsymbol{\xi})u(0,\boldsymbol{\xi})d\boldsymbol{\xi}, \qquad (7)$$

where $G(t,\boldsymbol{y}) = \frac{1}{(4\pi t)^{n/2}}e^{-|\boldsymbol{y}|^2/4t}$ *is the heat kernel.*

This theorem states, in particular, the convergence of the integral in the r.h.s. of (7).

Proof. In order to obtain (7) we show that two opposite inequalities hold in (7):

1. ($\geqslant$) For any fixed $A > 0$ consider the function:

$$v_A(t,\boldsymbol{x}) = u(t,\boldsymbol{x}) - \int_{|\boldsymbol{\xi}|<A} G(t,\boldsymbol{x}-\boldsymbol{\xi})u(0,\boldsymbol{\xi})d\boldsymbol{\xi}.$$

[1] for the case of constant coefficients with the proof for the 1D case only

It is sufficient to prove that $\forall\varepsilon > 0$ we have $v_A(t,\boldsymbol{x}) \geqslant -\varepsilon$. To show this we apply the maximum principle for function v_A in the cylinder $[0,T) \times \{|\boldsymbol{x}| \leqslant B\}$ for sufficiently large B, noting that, $v_A \geqslant -\varepsilon$ on the boundary.

2. ($\leqslant$) We have proved that the integral on the r.h.s. of (7) converges and, therefore, defines some function v such that $\bar{u} = u - v$ is a non-negative solution of the heat equation with zero initial state.

It remains to prove that a non-negative solution u of the heat equation with zero initial state is zero. Without loss of generality we can assume that $u_t \geqslant 0$. Indeed, otherwise we consider the function $\tilde{u}(t,\boldsymbol{x}) = \int_0^t u(\tau,\boldsymbol{x})d\tau$. [2] Now we prove that the function u satisfies the conditions of the Tikhonov uniqueness theorem (see. e.g., [7], section 3.4). Let $\delta > 0$ and $t + \delta < T$, then

$$u(t,\boldsymbol{x}) \overset{\Delta u \geqslant 0}{\leqslant} \frac{1}{\operatorname{Vol} B(\boldsymbol{x},|\boldsymbol{x}|)} \int_{B(\boldsymbol{x},|\boldsymbol{x}|)} u(t,\boldsymbol{\xi})d\boldsymbol{\xi} \leqslant C|\boldsymbol{x}|^{-n} \int_{B(\boldsymbol{0},2|\boldsymbol{x}|)} u(t,\boldsymbol{\xi})d\boldsymbol{\xi} \leqslant$$

$$\frac{C|\boldsymbol{x}|^{-n}}{G(\delta,2\boldsymbol{x})} \int_{\mathbb{R}^n} G(\delta,-\boldsymbol{\xi})u(t,\boldsymbol{\xi})d\boldsymbol{\xi} \overset{\substack{1^{\text{st}}\text{ part} \\ \text{of the proof}}}{\leqslant} \frac{C(4\pi\delta)^{n/2}}{|\boldsymbol{x}|^n} e^{|\boldsymbol{x}|^2/\delta} u(t+\delta,\boldsymbol{0}).$$

The last inequality, as well as the convergence of the integral, follows from the fact already proved that in (7) there is the inequality "$\geqslant$".

Since for any $\varepsilon > 0$ the function u is bounded in $[0, T-\varepsilon] \times \{|\boldsymbol{x}| \leqslant 1\}$ we see that with a suitable constant C_2 we have the inequality

$$u(t,\boldsymbol{x}) \leqslant C_2 e^{2|\boldsymbol{x}|^2/\varepsilon},$$

in the strip $[0,T-\varepsilon] \times \mathbb{R}^n$. □

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[2] to justify that $\tilde{u}$ satisfies the heat equation one, in particular, needs to exchange the derivatives with the time integration. For this, some extra regularity of u may be needed. It can be achieved by a mollification in the $\boldsymbol{x}$ variable, which preserves the heat equation.