Estimates for Spatial Derivatives of Solutions for Quasilinear Parabolic Equations with Small Viscosity

by

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Declaration

I hereby declare that the work presented in this thesis was carried out by myself at Heriot-Watt University, except where due acknowledgement is made, and has not been submitted for any other degree.

Signature of the candidate: Date:

Signature of the supervisor Professor S. Kuksin : Date:

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Abstract

The main subject of this thesis is the forced generalised Burgers equation

$$\partial_t \boldsymbol{u} + \nabla_{\boldsymbol{f}(\boldsymbol{u})}\boldsymbol{u} = \nu\Delta\boldsymbol{u} + \boldsymbol{h}(t,\boldsymbol{x})\,.$$

Here $\nabla_{\boldsymbol{f}(\boldsymbol{u})}\boldsymbol{u} \equiv \big(\boldsymbol{f}(\boldsymbol{u}),\nabla\big)\boldsymbol{u}$ stands for an arbitrary flow-type nonlinear term. The number of spatial dimensions is equal to n and $\dim \boldsymbol{u} = m$.

For space-periodic initial data we obtain upper and lower estimates for the Sobolev H^k norms of solutions in term of powers of the viscosity ν. This allows us to obtain some upper and lower estimates for the distribution of the Fourier coefficients (i.e. the spectrum) of solutions. The inequalities obtained in a sense allow to state a Kolmogorov-Obukhov type spectral law for the solutions of the generalised Burgers equation. We also obtain bounds for the Kolmogorov dissipation scale and exponent in the spectral law.

In Chapter 1 we consider general theorems concerning the Single conservation Law. In particularly we extend Lax's proof for the explicit construction of the solutions to a more general case.

In Chapter 2 we prove upper and lower estimates for Sobolev H^k norms of solutions the of 1D generalised Burgers equation. We note that in this case upper and lower estimates have the same order in terms of powers of the viscosity. This allows to determine exact parameters in the spectral law.

In Chapter 3 we consider the multidimensional case for which, in general, the number of space variables does not equal the number of unknown functions. As an applications of the results obtained we establish lower bounds for derivatives of solutions for the Navier-Stokes system.

Chapter 4 is devoted to upper and lower estimates for Sobolev H^k norms of the Non-linear Schrödinger equations.

Chapter 1

Single conservation law

1.1 Introduction

This chapter is devoted to the single conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0\,, \tag{1.1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a C^1-smooth function (the flux function). Some additional assumptions will be made subsequently.

The initial value problem (the Cauchy problem) consists of determining the solution u of (1.1.1) from its initial state $u(0,x) = \varphi(x)$ for all future time. We remark that the Cauchy problem for this equation is, in general, not solvable even in the class of continuous functions, even if φ is a C^∞-smooth function. For this reason it is normal to consider weak solutions in the sense of the corresponding integral identity.

We assume that the function $\varphi(\cdot)$ is in the space $L_\infty(\mathbb{R})$. Denote

$$\varphi_+ = \operatorname{ess\,sup} \varphi\,, \qquad \varphi_- = \operatorname{ess\,inf} \varphi\,. \tag{1.1.2}$$

Definition 1.1.1. *We say that a measurable function $u = u(t,x)$, bounded in any strip $[0,T] \times \mathbb{R}$, is a* weak *solution of the conservation law (1.1.1) with the initial state*

$$u(0,x) = \varphi(x) \in L_\infty(\mathbb{R})$$

if the following integral relation

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}\big(w_t u + w_x f(u)\big)dxdt + \int_{-\infty}^{+\infty} w(0,x)\varphi(x)dx = 0 \qquad (1.1.3)$$

is satisfied for all smooth test functions $w(t,x)$ vanishing for $|x|+t$ large enough.

However, this approach leads to loss of uniqueness of the solution of the initial value problem (see, e.g., [31]). We need to impose some additional condition which ensures uniqueness. Further on we consider viscosity solutions, entropy solutions and Oleinik-Lax solutions (in the sense of Oleinik-Lax variational principle).

For smooth solutions the notions of the weak solution and the pointwise solution of (1.1.1) are the same.

If u is a piecewise smooth weak solution then from the integral relation (1.1.3) it follows that u satisfies Rankine Hugoniot or jump relation ([31])

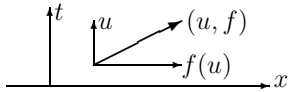
$$s[u] = [f]\ . \qquad (1.1.4)$$

Moreover a piecewise smooth function u satisfies the integral relation (1.1.3) iff u satisfies the jump condition (1.1.4) on the lines of discontinuity and satisfies equation (1.1.1) at all other points. Here [ ] denotes the jump across the line of discontinuity and s is the velocity of propagation of the discontinuity (or the shock). If the line of the shock is given by $x = x(t)$, then $s = \frac{dx}{dt}$.

Consider the vector field (u, f).

(1.1.1) means that this is a divergence-free vector field: $\mathrm{div}(u,f) = u_t + f_x = 0$.

(1.1.3) means that (u, f) is divergence-free in the generalised sense.



The jump relation (1.1.4) expresses the fact that the component of the vector field (u, f) in the direction normal to the line of discontinuity is continuous across the line of discontinuity.

This Chapter is organised as follows. In Section 1.2 we describe the general properties of weak solutions of the single conservation law. No additional assumptions are made here. (We only assume that the flux function $f \in C^1$, and the initial state

$\varphi \in L_\infty$.) In Section 1.3 for the case of a convex flux function f we introduce the notion of the entropy due to Oleinik. The entropy condition implies uniqueness of the weak solution. We also explain briefly why the inequality given by Oleinik is named an "entropy condition". In Section 1.4 we give an explicit construction for a weak solution due to Oleinik and Lax. We reproduce an elegant proof due to Lax ([23]) of the fact that the function constructed is a weak solution, filling in some details. Also our assumptions are weaker than Lax's original assumptions ([23]). We assume the *strict* convexity of the flux function f. This means that $f \in C^1$ and f' is a increasing function, and therefore f' is a one to one map between its range and the domain of definition. We remark that Lax has also described his assumption as one of strict convexity. However, expressed in our terms his assumption is one of uniform convexity on every bounded interval. (Uniform convexity means that $f \in C^1$ and there exist $\varepsilon > 0$ such that $f'(x) - \varepsilon x$ is an increasing function. For C^2-smooth functions this is equivalent to inequality $f'' \geqslant \varepsilon$.) For example, we can treat the case $f(u) = u^4$, but this case is not covered by Lax's paper [23]. Some applications are given. In Section 1.5 we study the inviscid limit for the periodic case and the rate of convergence as the viscosity tends to zero. No convexity of f is assumed here, but we assume smoothness for both the initial state and the flux function f.

In this chapter we summarise the background information needed for the second Chapter. For recent developments in Conservation law theory we refer to [4].

To conclude this introduction we note that the case of a concave flux function f is reduced to the convex case by the substitution $f(z) = -\tilde{f}(-z)$, $u(t,x) = -\tilde{u}(t,x)$.

1.2 General properties of weak solutions

We start this section with a lemma, which provides an equivalent definition of the notion of a weak solution, which will be used in the sequel:

Lemma 1.2.1. *Let $u(t,x)$ be a measurable function, bounded in any strip $[0,T]\times\mathbb{R}$. Then the integral identity (1.1.3) holds for any smooth test function $w(t,x)$ vanishing for $|x|+t$ large enough iff the following 2 conditions are holds:*

1. For any smooth test function w supported by a compact set in $\{t>0\}\times\mathbb{R}$ we have

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}\big(w_t u + w_x f(u)\big)dxdt = 0\,. \tag{1.2.1}$$

2. The function $\varphi(x)$ is the weak limit of $u(t,x)$ as $t\to 0$, i.e., for any smooth function $q(x)$ with bounded support there exists a set $\mathcal{E}\in[0,T]$ of Lebesgue measure 0 such that

$$\lim_{\substack{t\to 0\\ t\in[0,T]\setminus\mathcal{E}}}\int_{-\infty}^{+\infty}u(t,x)q(x)dx = \int_{-\infty}^{+\infty}\varphi(x)q(x)dx\,. \tag{1.2.2}$$

Before proving this Lemma we prove an auxiliary lemma

Lemma 1.2.2. *Take a real number A and real valued measurable functions $h(t)$ and $p(t)$ which are defined for $t\geqslant 0$ and belong to the space $L_{1,loc}$. Suppose that for any C^∞-smooth function $g(t)$ with bounded support we have*

$$\int_0^{+\infty}g'(t)h(t)dt + \int_0^{+\infty}g(t)p(t)dt + g(0)A = 0\,. \tag{1.2.3}$$

Then for almost all $t\geqslant 0$ we have

$$h(t) = A + \int_0^t p(\tau)d\tau\,.$$

Proof. We denote $P(t) = A + \int_0^t p(\tau)d\tau$. From the fact that $p\in L_{1,loc}$ it follows that the function $P(t)$ is absolutely continuous, differentiable almost everywhere and moreover $P'(t)=p(t)$ for almost all t (see [15]). Hence the function $g(t)P(t)$ is also an absolutely continuous function, and the identity

$$\tfrac{d}{dt}\big(g(t)P(t)\big) = g'(t)P(t) + g(t)p(t)$$

holds whenever $P'(t)=p(t)$. From this it follows that

$$\int_0^{+\infty}g'(t)P(t)dt + \int_0^{+\infty}g(t)p(t)dt + g(0)A = \int_0^{+\infty}\tfrac{d}{dt}\big(g(t)P(t)\big)dt + g(0)P(0) = 0.$$

Define the function $h_0(t)$ as $h_0(t) = h(t) - P(t)$. Substituting $h(t) = h_0(t) + P(t)$, into (1.2.3), we obtain

$$\int_0^{+\infty}g'(t)h_0(t)dt = 0$$

for any smooth function $g(t)$.

From this we have that $h_0(t)=0$ for a.a. t and the lemma follows. □

Proof of the Lemma 1.2.1. To prove the necessity we should derive (1.2.2) from (1.1.3), since condition (1.2.1) is already contained in condition (1.1.3). To derive (1.2.2) we consider relation (1.1.3) with the test function $w(t,x) = g(t)q(x)$. Then (1.1.3) becomes

$$\int_0^{\infty}g'(t)\int_{-\infty}^{+\infty}u(t,x)q(x)dxdt + \int_0^{\infty}g(t)\int_{-\infty}^{+\infty}f(u(t,x))q'(x)dxdt + g(0)\int_{-\infty}^{+\infty}u(0,x)q(x)dx = 0\,.$$

Now we can use Lemma 1.2.2 with

$$h(t) = \int_{-\infty}^{+\infty}u(t,x)q(x)dx, \qquad p(t) = \int_{-\infty}^{+\infty}f(u(t,x))q'(x)dx$$

and $A = \int_{-\infty}^{+\infty}u(0,x)q(x)dx$. (We note that both $h(t)$ and $p(t)$ are locally bounded functions, hence they are locally L_1-functions.) From Lemma 1.2.2 we have

$$h(t) = A + \int_0^t p(\tau)d\tau \quad \text{for a. a. } t.$$

Hence, up to a set of Lebesgue measure 0, the function $h(t)$ is a locally Lipschitz function, and $\operatorname{ess}\lim_{t\to 0}h(t) = A$. We arrive at (1.2.2).

Now let us prove the sufficiency, i.e., derive (1.1.3) from (1.2.1) and (1.2.2). Take any smooth function $w(t,x)$ with bounded support. Let $\chi(t)$ be a C^∞ smooth non-increasing function such that $\chi(t)\equiv 1$ for $t\in[0,\frac12]$ and $\chi(t)\equiv 0$ for $t>1$. Let us denote the left hand side of (1.1.3) as $I(w)$. Then we have

$$I(w) = \lim_{n\to\infty} I\big(\big(w(0,x) + (w(t,x)-w(0,x))\big)\chi(nt) + (1-\chi(nt))w(t,x)\big) = \lim_{n\to\infty} I\big(w(0,x)\chi(nt)\big) + \lim_{n\to\infty} I\big((w(t,x)-w(0,x))\chi(nt)\big) + \lim_{n\to\infty} I\big((1-\chi(nt))w(t,x)\big). \tag{1.2.4}$$

The third limit is equal to zero due to (1.2.1).

Consider the second limit.

$$I\big((w(t,x)-w(0,x))\chi(nt)\big) = \int_0^{+\infty}\int_{-\infty}^{+\infty} w_t(t,x)\chi(nt)u(t,x) + (w(t,x)-w(0,x))n\chi'(nt)u(t,x) + (w_x(t,x)-w_x(0,x))\chi(nt)f(u(t,x))dxdt = I_1 + I_2 + I_3.$$

The values I_1 and I_3 go to zero as n goes to infinity since the supports of there integrands shrink to zero while the integrands themselves remain bounded. For the value I_2 we have:

$$|I_2| \leqslant (X_+ - X_-) \int_0^\infty n\chi'(nt)dt \max|u| \max_{t\in[0,\frac{1}{n}],x\in\mathbb{R}} |w(t,x) - w(0,x)|.$$

Here $X_+ = \sup\{x : (t,x) \in \operatorname{supp} w\}$ and respectively $X_- = \inf\{x : (t,x) \in \operatorname{supp} w\}$. Hence I_2 goes to zero as $n \to \infty$, since $\int_0^\infty n\chi'(nt)dt = 1$ for all n, but $\max_{t\in[0,\frac{1}{n}],x\in\mathbb{R}} |w(t,x) - w(0,x)| \to 0$ because $w(t,x)$ is a smooth function with compact support.

Consider the first limit in the right hand side of (1.2.4).

$$I\big(w(0,x)\chi(nt)\big) = \int_0^{+\infty}\int_{-\infty}^{+\infty} w(0,x)n\chi'(nt)u(t,x)dxdt + \int_0^{+\infty}\int_{-\infty}^{+\infty} w_x(0,x)\chi(nt)f(u(t,x))dxdt + \int_{-\infty}^{+\infty} w(0,x)u(0,x)dx = \int_0^{+\infty}\Big(\int_{-\infty}^{+\infty} w(0,x)u(t,x)dx - \int_{-\infty}^{+\infty} w(0,x)u(0,x)dx\Big)n\chi'(nt)dt + \int_0^{+\infty}\int_{-\infty}^{+\infty} w_x(0,x)\chi(nt)f(u(t,x))dxdt = I_1 + I_2.$$

The value I_2 goes to zero as n goes to infinity since the supports of the integrand shrink to zero while the integrand itself remains bounded. For the value I_1 we have

$$|I_1| \leqslant \Big|\int_{-\infty}^{+\infty} w(0,x)u(t,x)dx - \int_{-\infty}^{+\infty} w(0,x)u(0,x)dx\Big|_{L_\infty(\{t:0\leqslant t\leqslant\frac{1}{n}\})} \Big|n\chi'(nt)\Big|_{L_1([0,+\infty])}.$$

The first factor tends to 0 as $n \to \infty$ due to (1.2.2), while the second factor always equals 1. □

The following lemma provides a definition of the stream function of a solution of single conservation law. For the sake of brevity in this lemma $f(t,x)$ denotes $f(u(t,x))$.

Lemma 1.2.3. *Suppose that u is a weak solution of the single conservation law (1.1.1) in the sense of definition 1.1.1. Then there exists a unique function $U(t,x)$ (the stream function) defined on $[0,+\infty)\times\mathbb{R}$, which is locally absolutely continuous w.r.t. each variable such that*

$$U_x = u \qquad \text{almost everywhere,} \tag{1.2.5}$$

$$U_t = -f \qquad \text{almost everywhere,} \tag{1.2.6}$$

$$U(0,0) = 0. \tag{1.2.7}$$

This function U satisfies a global Lipschitz condition in every strip $\Pi_T = [0,T]\times\mathbb{R}$: For every pair (t_1,x_1) and (t_2,x_2) from Π_T we have

$$|U(t_2,x_2) - U(t_1,x_1)| \leqslant |t_2 - t_1| \max_{\Pi_T} f + |x_2 - x_1| \max_{\Pi_T} u. \tag{1.2.8}$$

Also this function satisfies

$$U(0,x) = \int_0^x \varphi(\xi)d\xi. \tag{1.2.9}$$

Proof. Uniqueness. First we recall that any locally absolutely continuous function $h(s)$ is differentiable almost everywhere, the derivative belongs to the space $L_{1,loc}$ and the Newton-Leibniz formula holds:

$$h(s_2) - h(s_1) = \int_{s_1}^{s_2} h'(s)ds.$$

Suppose that U_1 and U_2 are two functions that are locally absolutely continuous w.r.t. each variable and satisfy (1.2.5), (1.2.6), (1.2.7). Then the function $\widetilde{U} = U_2 - U_1$ satisfies

$$\widetilde{U}_x = 0 \qquad \text{almost everywhere,} \tag{1.2.10}$$

$$\widetilde{U}_t = 0 \qquad \text{almost everywhere,} \tag{1.2.11}$$

$$\widetilde{U}(0,0) = 0. \tag{1.2.12}$$

From (1.2.10) it follows that for almost all t $U(t,x)$ does not depend on x. (Since if some condition holds for almost all (t,x), then for almost all t it holds for almost all x and vice versa.) Similarly (1.2.11) implies that for almost all x $U(t,x)$ does not depend on t. Since $U(t,x)$ is a locally absolutely continuous w.r.t. each variable and hence continuous w.r.t. each variable we conclude that $\tilde{U}$ is a constant function. This constant is zero by (1.2.12).

The existence of the stream function follows from a generalisation of the Weyl decomposition:

Suppose that Ω is a bounded domain in $\mathbb{R}^m$ with C^1-smooth boundary. Let $\boldsymbol{u}(\boldsymbol{x}) \in L_q(\Omega, \mathbb{R}^m)$ with $q \geqslant 1$ and suppose that for any divergence-free smooth vector valued function $\boldsymbol{v}$ supported by a compact set in Ω we have

$$\int_\Omega (\boldsymbol{u}, \boldsymbol{v})_m dx = 0. \tag{1.2.13}$$

Then there exists a function $p \in W_q^1(\Omega)$ such that $\boldsymbol{u} = \nabla p$. (See [28]).

We use this theorem with $m = 2$, $\boldsymbol{u} = \binom{f}{-u}$, $\binom{x_1}{x_2} = \binom{t}{x}$, noting that for any divergence free smooth vector field $\binom{v_1}{v_2}$ with bounded support there exists a smooth function w with bounded support such that $\binom{v_1}{v_2} = \binom{w_x}{-w_t}$. Indeed, set $w(t,x) = \int_A^{(t,x)} v_1 dx - v_2 dt$. Choosing the the first point of integration (A) far enough from the support of $\boldsymbol{v}$, we obtain that w also has a bounded support since this integral does not depend on the path of integration by the Stokes' theorem. Now (1.2.13) follows from (1.1.3). We set $U(t,x) = -p(t,x)$. Let us show that the function U, up to a set of of Lebesgue measure 0, is a Lipschitz function and (1.2.8) holds. Indeed, consider the mollifications $U^\varepsilon = U * \pi^\varepsilon$, where $\pi^\varepsilon(x) = \frac{1}{\varepsilon^2}\pi^\varepsilon(\frac{x}{\varepsilon})$ and π is a standard molifier in $\mathbb{R}^2$. For each U^ε (1.2.8) holds. Using the Ascoli-Arzelà Theorem we conclude that U^ε uniformly converge to its limit as $\varepsilon \to 0$. Hence the limit satisfies (1.2.8).

The relation (1.2.9) follows from (1.2.8) and Lemma 1.2.2 due to the uniqueness of the weak limit. □

Lemma 1.2.4. *Suppose that the function $U(t,x)$ belongs $Lip([0,T] \times \mathbb{R})$ for any T and that the relation*

$$U_t + f(U_x) = 0 \tag{1.2.14}$$

holds for almost all (t,x) in $\{t \geqslant 0\} \times \mathbb{R}$. Then the function $u(t,x) = \frac{\partial}{\partial x} U(t,x)$ is a weak solution of the single conservation law (1.1.1) in the sense of (1.1.3) with the initial state $\varphi(x) = U_x(0,x)$.

Proof. Let $w = w(t,x)$ be a smooth function with bounded support. We multiply (1.2.14) by w_x and integrate over the half-plane $\{t \geqslant 0\}$. Integrating by parts we arrive at (1.1.3). □

1.3 The notion of entropy

Definition 1.3.1. *Let $u = u(t,x)$ be a measurable weak solution of the single conservation law of (1.1.1) with $f'' \geqslant 0$ defined on $[0, +\infty) \times \mathbb{R}$. We say that the function u satisfies the entropy condition if for any bounded domain D there exists a non-increasing function $K(t)$ defined for $t > 0$ such that for any pair of points (t, x_1) and (t, x_2) from the domain D and $t > 0$, we have:*

$$\frac{u(t,x_2) - u(t,x_1)}{x_2 - x_1} \leqslant K(t)\,. \tag{1.3.1}$$

We call any solution that satisfies an entropy condition an entropy solution.

Theorem 1.3.1. (Oleinik, [26]). *Suppose u_1 and u_2 are two entropy weak solutions of the Cauchy problem for (1.1.1) with equals initial data with C^2-smooth f and $f'' \geqslant 0$. Then $u_1(t,x) = u_2(t,x)$ for almost all $(t,x) \in \{t \geqslant 0\}$.*

Remark. In fact, article [26] considers the case of a C^2-smooth function $f = f(t,x,u)$ such that $f''_{uu} \geqslant 0$ and $f'_u(t,x,u)$ is bounded whenever u is bounded, $t \geqslant 0$, and $x \in \mathbb{R}$.

Proposition. *Suppose that the function f in (1.1.1) is C^2-smooth and uniformly convex, i.e.,*

$$f'' \geqslant \varepsilon > 0 \tag{1.3.2}$$

and the initial state φ is a bounded measurable function. Then there exists a weak solution of the Cauchy problem such that

$$\frac{u(t,x_2) - u(t,x_1)}{x_2 - x_1} \leqslant \frac{1}{\varepsilon t}\,, \qquad \text{for all } t > 0 \text{ and } x_1 \neq x_2 \in \mathbb{R}\,. \tag{1.3.3}$$

For the proof see the next section.

Broadly speaking, condition (1.3.3) means that $u_x \in [-\infty, \frac{1}{\varepsilon t}]$ including minus infinity. Let us give a heuristic derivation of this condition. We note that for smooth solutions there is a relation $u(t,x) = \varphi(x - tf'(u(t,x)))$. From the implicit function theorem it follows that $u_x = \frac{\varphi'}{1+\varphi' f'' t}$. Thus, if $\varphi' \leqslant 0$ then $u_x \leqslant 0$. The denominator is greater then 0 because of the smoothness of the function u. If $\varphi' > 0$ then $u_x \leqslant \frac{\varphi'}{\varphi' f'' t} \leqslant \frac{1}{\varepsilon t}$.

Remark. For the existence of an entropy weak solution of the Cauchy problem for (1.1.1) condition (1.3.2) can be relaxed to $f'' > 0$. Indeed, the positivity of the second derivative implies that f is uniformly convex on every bounded interval. (Here we assume that f'' is a continuous function.)

Proposition. *Suppose that f is C^1-smooth such that f' is an increasing function and the initial state φ satisfies the condition*

$$\frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \leqslant K\ , \qquad \text{for all } x_1 \neq x_2 \in \mathbb{R}\,. \tag{1.3.4}$$

Then there exists a weak solution of the Cauchy problem such that

$$\frac{u(t, x_2) - u(t, x_1)}{x_2 - x_1} \leqslant K\ , \qquad \text{for all } t > 0 \text{ and } x_1 \neq x_2 \in \mathbb{R}\,. \tag{1.3.5}$$

For the proof see the next section.

If both (1.3.2) and (1.3.4) do not hold then we can not guarantee the existence of a weak entropy solution.

Example. Consider the Cauchy problem

$$\begin{cases} u_t + (\frac{1}{4}u^4)_x = 0, \\ u(0, x) = \sqrt[3]{x}. \end{cases} \tag{1.3.6}$$

This problem has a natural continuous weak solution

$$u(t, x) = \sqrt[3]{\tfrac{x}{1+t}}\,. \tag{1.3.7}$$

This solution is a solution in all senses other than entropy. In the next section (see Lemma 1.4.18) we show that the Cauchy problem (1.3.6) has no entropy weak solutions at all.

We remark that there is a misprint in [26]. To obtain the existence of an *entropy* weak solution we should demand $f''_{uu} > 0$, not $f''_{uu} \geqslant 0$. While the uniqueness theorem holds under the assumption $f''_{uu} \geqslant 0$.

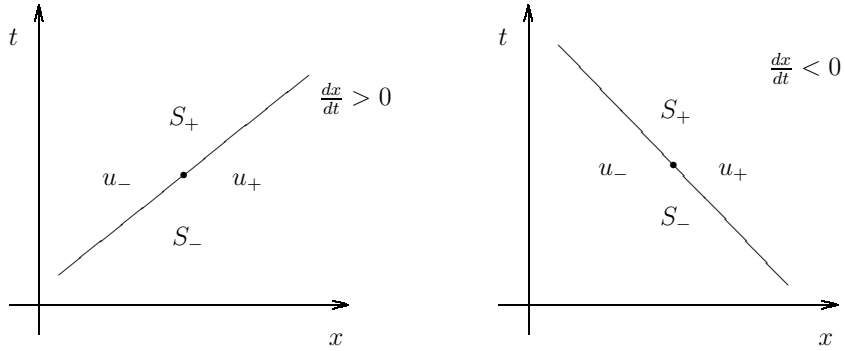
Why is condition (1.3.1) called an entropy condition (moreover, a 'property of nondecreasing entropy')? The reason is that real physical processes are irreversible

and the function that represent this irreversibility is called the entropy. Consider the function

$$S(t, x) = f(u(t, x)).$$

Let us show that condition (1.3.1) implies that the function $S(t, x)$ is non-decreasing in time across any line of discontinuity Γ (the shock line):

$$S_+ = S(t + 0, x) \geqslant S_- = S(T - 0, x), \quad (t, x) \in \Gamma = \{(t, x) : x = x(t)\}.$$



From (1.3.1) we have $u_- \geqslant u_+$. Using the Rankine Hugoniot relation

$$\tfrac{dx}{dt}(u_+ - u_-) = f(u_+) - f(u_-)$$

we see that:

$$\text{if } \tfrac{dx}{dt} > 0 \text{ then } S_- = f(u_+) < f(u_-) = S_+;$$

$$\text{if } \tfrac{dx}{dt} < 0 \text{ then } S_+ = f(u_+) > f(u_-) = S_-.$$

1.4 Explicit Solution

In this section we assume that the flux function f in (1.1.1) is *strictly* convex, i.e., f is C^1-smooth and f' is a increasing function. This implies that f' is a one to one map between its range and the domain of definition. We assume that the initial state φ is an arbitrary function from the space $L_\infty(\mathbb{R})$.

Let the function $b(\cdot)$ be the inverses of the function $f'(\cdot)$, i.e. the domain of $b(\cdot)$ is the range of the function $f'(\cdot)$ and $b\big(f'(u)\big) = u$ and $f'\big(b(s)\big) = s$.

Clearly $b(\cdot)$ is a continuous increasing function and that $b(s)$ tends to infinity as s approaches the endpoints of the domain of definition of b.

We define a function $g(\cdot)$ by the formula

$$g(s) = sb(s) - f(b(s)) \tag{1.4.1}$$

The function $g(\cdot)$ is in fact the conjugate (or Legendre transformation) of f $\big(g(s) = \max_u(us - f(u))\big)$, although we shall not use this fact.

We note that g is also a C^1-smooth convex function and

$$g'(s) = b(s). \tag{1.4.2}$$

Indeed, if the function b is a differentiable function then (1.4.2) is obtained by differentiation of (1.4.1). In the general case we should consider the finite difference

$$g(s+\triangle s) - g(s) = (s+\triangle s)b(s+\triangle s) - s\cdot b(s) - \big(f(b(s+\triangle s)) - f(b(s))\big) = \\ \triangle s\, b(s+\triangle s) + s\cdot\big(b(s+\triangle s) - b(s)\big) - f'(\theta)\big(b(s+\triangle s) - b(s)\big),$$

where $\theta \in (b(s), b(s+\triangle s))$, so $f'(\theta) \in (s, s+\triangle s)$. We obtain the formula:

$$\frac{g(s+\triangle s) - g(s)}{\triangle s} = b(s+\triangle s) + \varkappa\cdot\big(b(s+\triangle s) - b(s)\big).$$

Here $\varkappa = \varkappa(\triangle s) \in (0,1)$. Passing $\triangle s$ to zero and using the continuity of the function b, we arrive at (1.4.2).

Consider a primitive function of the initial state:

$$\Phi(y) = \int_0^y \varphi(\eta)d\eta\,. \tag{1.4.3}$$

From (1.1.2) it follows that $\Phi(\cdot)$ is a Lipschitz function and the upper and the lower Lipschitz constants for this function are φ_+ and φ_-, respectively, i.e.,

$$\varphi_- \leqslant \tfrac{\Phi(x_2)-\Phi(x_1)}{x_2-x_1} \leqslant \varphi_+\,. \tag{1.4.4}$$

Consider the auxiliary function

$$\psi(t,x,y) = \psi_{t,x}(y) = \Phi(y) + t\cdot g\left(\tfrac{x-y}{t}\right). \tag{1.4.5}$$

For fixed x and t this is a continuous function of y, which is increasing in a neighbourhood of the right endpoint of its domain of definition and decreasing in a neighbourhood of the right endpoint. Indeed, the second term in the r.h.s. of (1.4.5) has a

continuous first derivative, which tends to plus or minus infinity as y approaches the right or left endpoints of its domain of definition, respectively. On the other hand, the first term is a Lipschitz function due to (1.4.4) and the first derivative is bounded. Therefore $\psi_{t,x}(y)$ has a finite global minimum, and there is non-empty set of points $\{y\}$, contained in the interior of the domain of definition, at which this minimum is achieved. We refer to this set as the set of minimum points.

Indeed, consider the interval of y where the derivative of the function $y \mapsto t\cdot g\left(\frac{x-y}{t}\right)$ w.r.t. y is less than $-\varphi_+$, i.e.,

$$\mathcal{O}^{\text{left}}_{t,x} = \left\{y : -b\left(\tfrac{x-y}{t}\right) < -\varphi_+\right\}. \tag{1.4.6}$$

Since the function $y \mapsto -b\left(\frac{x-y}{t}\right)$ is an increasing function and goes from minus infinity to plus infinity, this is a (semi) neighbourhood of the left endpoint of the domain of definition of the function $\psi_{t,x}(y)$. Since the functions b and f' are inverses each of other, we have:

$$\mathcal{O}^{\text{left}}_{t,x} = \{x - tf'(+\infty) < y < x - tf'(\varphi_+)\}\,.$$

Combining (1.4.4) and the inequality in (1.4.6), we get that the function $\psi_{t,x}(y)$ is decreasing on this neighbourhood. Similarly, we consider the "opposite" interval of y:

$$\mathcal{O}^{\text{right}}_{t,x} = \left\{y : -b\left(\tfrac{x-y}{t}\right) > -\varphi_-\right\} = \{x - tf'(\varphi_-) < y < x - tf'(-\infty)\}\,, \tag{1.4.7}$$

and conclude that the function $\psi_{t,x}(y)$ is increasing on this neighbourhood of the right endpoint of its domain of definition.

We denote

$$A_{t,x} = x - tf'(\varphi_+) \qquad B_{t,x} = x - tf'(\varphi_-)\,. \tag{1.4.8}$$

We have proved that if y_{min} is any one of the minimum points for the function $\psi_{t,x}(y)$, then

$$y_{min} \in [A_{t,x}, B_{t,x}] \tag{1.4.9}$$

and hence

$$\varphi_- \leqslant b\left(\tfrac{x-y_{min}}{t}\right) \leqslant \varphi_+\,. \tag{1.4.10}$$

Similarly, considering the points

$$A'_{t,x} = x - tf'\Big(\max_{[A_{t,x},B_{t,x}]}\varphi\Big) \qquad B'_{t,x} = x - tf'\Big(\min_{[A_{t,x},B_{t,x}]}\varphi\Big)$$

we see that the function $\psi_{t,x}(\cdot)$ is decreasing on $(A_{t,x}, A'_{t,x})$ and is increasing on $(B'_{t,x}, B_{t,x})$.

Hence we have

$$y_{min} \in [A'_{t,x}, B'_{t,x}] \tag{1.4.11}$$

or, equivalently

$$b\left(\tfrac{x-y_{min}}{t}\right) \in \left[\min_{[A_{t,x},B_{t,x}]}\varphi\,,\ \max_{[A_{t,x},B_{t,x}]}\varphi\right]. \tag{1.4.12}$$

Lemma 1.4.1. *For fixed (t,x) the set of minimum points for the function (1.4.5) is independent of the function $f(\cdot)$ outside the interval $[\varphi_-, \varphi_+]$, provided f remains strictly convex.*

Proof. When we change f in this way, the functions $b(s)$ and $g(s)$ remain unchanged on the interval $s \in [f'(\varphi_-), f'(\varphi_+)]$, so the function $\psi_{t,x}(y)$ does not change its value if y belongs to the interval $\left\{y : \frac{x-y}{t} \in [f'(\varphi_-), f'(\varphi_+)]\right\} = \left\{A_{t,x} \leqslant y \leqslant B_{t,x}\right\}$. The proof concludes with (1.4.9) which states that all minimum points for this function are in this interval. □

Following Hopf [12] we state two lemmas.

Lemma 1.4.2. *Let y_1 be (any) value of y where the function ψ assumes its minimum for the parameters t, x_1, and, respectively, y_2 be (any) value of y where the function ψ assumes its minimum for the parameters t, x_2. If $x_1 < x_2$ then $y_1 \leqslant y_2$.*

Proof. By the definition of y_1 and y_2 as minimum points, we have

$$\psi_{t,x_1}(y_1) \leqslant \psi_{t,x_1}(y_2)\,,$$
$$\psi_{t,x_2}(y_2) \leqslant \psi_{t,x_2}(y_1)\,.$$

Adding these two inequalities we get

$$g\left(\tfrac{x_1-y_1}{t}\right) + g\left(\tfrac{x_2-y_2}{t}\right) \leqslant g\left(\tfrac{x_1-y_2}{t}\right) + g\left(\tfrac{x_2-y_1}{t}\right).$$

Now, in relation to the convex function $g(\cdot)$ we use the following fact:

If a) the mid-points of the intervals $[a_1, a_2]$ and $[b_1, b_2]$ coincide, i.e., $a_1 + a_2 = b_1 + b_2$, and b) $g(a_1) + g(a_2) \leqslant g(b_1) + g(b_2)$; then the first interval is contained in the second interval or equivalently, is no greater than the second interval, i.e., $|a_2 - a_1| \leqslant |b_2 - b_1|$.

We obtain:

$$|(x_2 - x_1) - (y_2 - y_1)| \leqslant |(x_2 - x_1) + (y_2 - y_1)|.$$

Since $x_2 - x_1 > 0$, this inequality is equivalent to the inequality $y_2 - y_1 \geqslant 0$. □

The following Lemma immediately follows from Lemma 1.4.2.

Lemma 1.4.3. *For given t, with the exception of a denumerable set of values of x, the function $\psi_{t,x}(\cdot)$ assumes its minimum at a single point which we denote by $y_0(t,x)$.*

With the help of y_0 we define the function

$$u(t,x) = b\left(\tfrac{x-y_0(t,x)}{t}\right). \tag{1.4.13}$$

This function is a weak solution of the Cauchy problem for (1.1.1). Before proving this, we establish some properties of the function (1.4.13).

Lemma 1.4.4. *Function (1.4.13) does not depend on the behaviour of the function f outside the interval $[\varphi_-, \varphi_+]$, provided it remains a strictly convex function.*

Proof. Due to Lemma 1.4.1, y_0 does not change its value. Due to (1.4.10) the value $\frac{x-y_0(t,x)}{t} \in [f'(\varphi_-), f'(\varphi_+)]$. The function $b(\cdot)$ on the interval $[f'(\varphi_-), f'(\varphi_+)]$ is independent of the behaviour of f outside the interval $[\varphi_-, \varphi_+]$ since it is the inverse of the monotonic function f'. □

Due to this Lemma, without loss of generality we can replace the function $f(\cdot)$ with the function

$$\tilde{f}(x) = \begin{cases} f(\varphi_+) + f'(\varphi_+)(x-\varphi_+) + \frac{a}{2}(x-\varphi_+)^2 & \text{for } x > \varphi_+, \\ f(x) & \text{for } x \in [\varphi_-, \varphi_+], \\ f(\varphi_-) + f'(\varphi_-)(x-\varphi_-) + \frac{a}{2}(x-\varphi_-)^2 & \text{for } x < \varphi_-. \end{cases} \tag{1.4.14}$$

Here a is a positive constant, e.g., $a = 1$. Accordingly the function b will be replaced with

$$\tilde{b}(s) = \begin{cases} \varphi_+ + \frac{1}{a}(s - f'(\varphi_+)) & \text{for } s > f'(\varphi_+) \\ b(s) & \text{for } s \in [f'(\varphi_-), f'(\varphi_+)] \\ \varphi_- + \frac{1}{a}(s - f'(\varphi_-)) & \text{for } s < f'(\varphi_-) \end{cases} \tag{1.4.15}$$

and the function g will be replaced with

$$\tilde{g}(s) = \begin{cases} g(f'(\varphi_+)) + \varphi_+(s - f'(\varphi_+)) + \frac{1}{2a}(s - f'(\varphi_+))^2 & \text{for } s > f'(\varphi_+) \\ g(s) & \text{for } s \in [f'(\varphi_-), f'(\varphi_+)] \\ g(f'(\varphi_-)) + \varphi_-(s - f'(\varphi_-)) + \frac{1}{2a}(s - f'(\varphi_-))^2 & \text{for } s > f'(\varphi_-) \end{cases} \tag{1.4.16}$$

In what follows we drop the tildes from $\tilde{f}$, $\tilde{b}$, $\tilde{g}$ and assume that f, b, g are given by (1.4.14), (1.4.15), (1.4.16), respectively.

Let us consider the sequence of probability measures with densities

$$\rho_N(y) = \frac{\exp\big(-N\psi_{t,x}(y)\big)}{\int_{-\infty}^{+\infty} \exp\big(-N\psi_{t,x}(y)\big)dy}\,. \tag{1.4.17}$$

The word "probability" here means only that the densities $\rho_N(y)$ are non-negative functions and the corresponding measures of the whole real line $\mathbb{R}$ are equal to 1 $\big(\int_{-\infty}^{+\infty} \rho_N(y)dy = 1\big)$.

Lemma 1.4.5. *For any continuous function $h(y)$ whose growth at infinity is strictly less than exponential, we have:*

$$\lim_{N\to\infty} h(y)\rho_N(y)dy = h(y_0)\ , \tag{1.4.18}$$

where y_0 is defined in Lemma 1.4.3 and the densities ρ_N are defined in (1.4.17).

Scheme of the proof. Adding a constant to the function ψ in (1.4.17) does not change the value of $\rho_N(y)$, so without loss of generality we can assume that the minimum value of the function ψ is equal to zero. Therefore the value of the numerator at the point y_0 always remains equal to 1 and at other points tends to 0 exponentially, so the functions $\rho_N(y)$ tends to the δ-function at the point y_0. □

We need calculate the derivatives of the function ψ:

$$\tfrac{\partial}{\partial x}\psi(t,x,y) = b\left(\tfrac{x-y}{t}\right)\ , \tag{1.4.19}$$

$$\tfrac{\partial}{\partial t}\psi(t,x,y) = g\left(\tfrac{x-y}{t}\right) + t\cdot g'\left(\tfrac{x-y}{t}\right)\cdot\tfrac{-(x-y)}{t^2} = g\left(\tfrac{x-y}{t}\right) - b\left(\tfrac{x-y}{t}\right)\tfrac{x-y}{t} \overset{(1.4.1)}{=} -f\Big(b\left(\tfrac{x-y}{t}\right)\Big)\ . \tag{1.4.20}$$

Theorem 1.4.6. (Lax [23]). *Consider the function $u(t,x)$ defined by (1.4.13). Then the vector field $\big(u, f(u)\big)$ is divergent-free in the generalise sense, i.e.,*

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} \big(w_t u + w_x f(u)\big)dxdt = 0\,, \tag{1.4.21}$$

for every C^1-smooth test function w, supported by a bounded compact set in $\{t > 0\}$. The function y_0 is defined in Lemma 1.4.3.

Remark. This Theorem is actually a particular case of the results of Oleinik's papers [25] and [27], where more general situation was studied. In these papers the constructions of the solution involved the minimisation of some functional. This functional is a generalisation of the function ψ (1.4.5). This method was subsequently named the Oleinik-Lax variational principle. Lax give a very elegant proof which we will now reproduce.

Proof. We can write $u(t,x)$ as $\lim\limits_{N\to\infty} u_N(t,x)$, where

$$u_N(t,x) = \frac{\int_{-\infty}^{+\infty} b(\frac{x-y}{t})\exp\big(-N\psi_{t,x}(y)\big)dy}{\int_{-\infty}^{+\infty}\exp\big(-N\psi_{t,x}(y)\big)dy}\ . \tag{1.4.22}$$

Denote the denominator in (1.4.22) by

$$V_N(t,x) = \int_{-\infty}^{+\infty}\exp\big(-N\psi_{t,x}(y)\big)dy\ . \tag{1.4.23}$$

In view of (1.4.19) we can write

$$u_N = \frac{\frac{1}{-N}(V_N)_x}{V_N} = \left(\frac{-1}{N}\log V_N\right)_x = (U_N)_x\ . \tag{1.4.24}$$

Likewise, we can write $f(u)$ as $\lim\limits_{N\to\infty} f_N$, where

$$f_N(t,x) = \frac{\int_{-\infty}^{+\infty} f\left(b(\frac{x-y}{t})\right)\exp\big(-N\psi_{t,x}(y)\big)dy}{V_N}\ . \tag{1.4.25}$$

Using (1.4.20) we get

$$f_N = \frac{\frac{1}{N}(V_N)_t}{V_N} = \left(\frac{1}{N}\log V_N\right)_t = -\,(U_N)_t\ . \tag{1.4.26}$$

From (1.4.24) and (1.4.26) we conclude that the vector fields (u_N, f_N) are divergence-free. Therefore, their (almost everywhere) pointwise limit $(u, f(u))$ is divergence-free in the generalised sense.

Indeed, the pair of functions (u_N, f_N) satisfies

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}\big(w_t u_N + w_x f_N\big)dxdt = 0\,, \tag{1.4.21$'$}$$

for every C^1-smooth test function w, supported by a bounded compact set in $\{t>0\}$. Since $(u, f(u))$ is the (almost everywhere) pointwise limit of the sequence (u_N, f_N) then (1.4.21) follows from (1.4.21$'$) by Lebesgue's bounded convergence theorem, provided we know, that (u_N, f_N) are locally uniformly bounded. The rest of the proof is devoted to show this.

To simplify the proof of the local uniform boundedness of the sequence (u_N, f_N), we assume that the constant a in (1.4.14), (1.4.15), (1.4.16) is equal to 1. Under this assumption we will show:

$$|u_N(t,x)| < |\varphi|_{L_\infty} + 1 + \tfrac{|\varphi_-|+|\varphi_+|+1}{\exp(\frac{tN}{2})-1} \tag{1.4.27}$$

and

$$|f_N(t,x)| < \max_{[\varphi_--1,\varphi_++1]}|f| + \tfrac{2|f(\varphi_-)|+2|f(\varphi_+)|+|f'(\varphi_-)|+|f'(\varphi_-)|+1+\frac{2}{Nt}}{2\exp(\frac{tN}{2})-2}\,. \tag{1.4.28}$$

We write $u_N(t,x)$ as

$$u_N(t,x) = \int_{-\infty}^{A_{t,x}-t} + \int_{A_{t,x}-t}^{B_{t,x}+t} + \int_{B_{t,x}+t}^{+\infty} b\left(\tfrac{x-y}{t}\right)\rho_N(y)\,dy = I_1 + I_2 + I_3\,.$$

Here $\rho_N(\cdot)$ is defined by (1.4.17); $A_{t,x}$ and $B_{t,x}$ are defined in (1.4.8). Clearly that

$$|I_2| \leqslant \max_{[f'(\varphi_-)-1, f'(\varphi_+)+1]}|b(\cdot)| = |\varphi|_{L_\infty} + 1,$$

since $\int_{-\infty}^{-\infty}\rho_N(y)\,dy = 1$.

Let us prove that

$$|I_3| < \tfrac{|\varphi_-|+\frac{1}{2}}{\exp(\frac{tN}{2})-1}\,. \tag{1.4.29}$$

First we show that if $B_{t,x} < c_1 < c_2$, then

$$\psi_{t,x}(c_2) - \psi_{t,x}(c_1) \geqslant \tfrac{t}{2}\left(\tfrac{B_{t,x}-c_2}{t}\right)^2 - \tfrac{t}{2}\left(\tfrac{B_{t,x}-c_1}{t}\right)^2. \tag{1.4.30}$$

Indeed, since $\frac{B_{t,x}-c_i}{t} < f'(\varphi_-)$ for $i=1,2$, then from (1.4.16) we have:

$$\psi_{t,x}(c_i) = \Phi(c_i) + t\left(g(f'(\varphi_-)) + \varphi_-(\tfrac{x-c_i}{t} - f'(\varphi_-)) + \tfrac{1}{2}(\tfrac{x-c_i}{t} - f'(\varphi_-))^2\right).$$

Since $\Phi(c_2) - \Phi(c_1) \geqslant \varphi_-(c_2-c_1)$, then (1.4.30) follows.

From (1.4.30) it follows that

$$\psi_{t,x}(y) \geqslant \psi_{t,x}(B_{t,x}+t) - \tfrac{t}{2} + \tfrac{t}{2}\left(\tfrac{B_{t,x}-y}{t}\right)^2 \qquad \text{if } y > B_{t,x}+t\,,$$

and

$$\psi_{t,x}(y) \leqslant \psi_{t,x}(B_{t,x}+t) - \tfrac{t}{2} + \tfrac{t}{2}\left(\tfrac{B_{t,x}-y}{t}\right)^2 \qquad \text{if } y \in (B_{t,x}, B_{t,x}+t)\,.$$

Hence

$$|I_3| < \frac{\int_{B_{t,x}+t}^{+\infty}\left|b(\frac{x-y}{t})\right|\exp(-N\psi_{t,x}(y))\,dy}{\int_{B_{t,x}}^{B_{t,x}+t}\exp(-N\psi_{t,x}(y))\,dy} \leqslant \frac{\int_1^{+\infty}|\varphi_- - z|\exp(-\frac{Nt}{2}z^2)\,dz}{\int_0^1\exp(-\frac{Nt}{2}z^2)\,dz}\,. \tag{1.4.31}$$

Here we also have made the substitution $z = \frac{y-B_{t,x}}{t}$. Now, using the inequalities and the identities:

$$\int_1^{+\infty}\exp(-\tfrac{Nt}{2}z^2)dz \leqslant \int_1^{+\infty}\exp(-\tfrac{Nt}{2}z)dz = \tfrac{2}{Nt}\exp(-\tfrac{Nt}{2})\,, \tag{1.4.32}$$

$$\int_0^1\exp(-\tfrac{Nt}{2}z^2)dz \geqslant \int_0^1\exp(-\tfrac{Nt}{2}z)dz = \tfrac{2}{Nt}(1-\exp(-\tfrac{Nt}{2}))\,, \tag{1.4.33}$$

$$\int_1^{+\infty} z\exp(-\tfrac{Nt}{2}z^2)dz = \tfrac{1}{Nt}\exp(-\tfrac{Nt}{2})\,, \tag{1.4.34}$$

we continue (1.4.31) with

$$|I_3| \leqslant \frac{\int_1^{+\infty}\left(|\varphi_-|\exp(-\frac{Nt}{2}z) + z\exp(-\frac{Nt}{2}z^2)\right)dz}{\int_0^1\exp(-\frac{Nt}{2}z)\,dz} = \frac{|\varphi_-|+\frac{1}{2}}{\exp(\frac{tN}{2})-1}\,.$$

We arrive at (1.4.29). Similarly, we have

$$|I_1| < \tfrac{|\varphi_+|+\frac{1}{2}}{\exp(\frac{tN}{2})-1}\,,$$

and (1.4.27) follows.

The proof of (1.4.28) is similar:

we write $f_N(t,x)$ as

$$f_N(t,x) = \int_{-\infty}^{A_{t,x}-t} + \int_{A_{t,x}-t}^{B_{t,x}+t} + \int_{B_{t,x}+t}^{+\infty} f\Big(b\left(\tfrac{x-y}{t}\right)\Big)\rho_N(y)\,dy = I_1 + I_2 + I_3\,.$$

Using the fact that $\int_{-\infty}^{+\infty}\rho_N(y)\,dy = 1$, we obtain

$$|I_2| \leqslant \max_{[f'(\varphi_-)-1, f'(\varphi_+)+1]} |f(b(\cdot))| = \max_{[\varphi_- -1, \varphi_+ +1]} |f(\cdot)|\,.$$

We have

$$|I_3| < \frac{\int_1^{+\infty}\left|f(\varphi_-) - zf'(\varphi_-) + \frac{1}{2}z^2\right|\exp(-\frac{Nt}{2}z^2)\,dz}{\int_0^1 \exp(-\frac{Nt}{2}z^2)\,dz}.$$

Using (1.4.32), (1.4.33), (1.4.34), and the inequality

$$\int_1^{+\infty} z^2\exp(-\tfrac{Nt}{2}z^2)dz = \tfrac{1}{2}\int_1^{+\infty} z\exp(-\tfrac{Nt}{2}z^2)dz^2 = \tfrac{1}{2}\int_1^{+\infty}\sqrt{y}\exp(-\tfrac{Nt}{2}y)dy \leqslant$$

$$\leqslant \tfrac{1}{2}\int_1^{+\infty} y\exp(-\tfrac{Nt}{2}y)dy = \left(\tfrac{1}{Nt} + \tfrac{2}{(Nt)^2}\right)\exp(-\tfrac{Nt}{2})\,,$$

we obtain

$$|I_3| < \frac{2\,|f(\varphi_-)| + |f'(\varphi_-)| + \frac{1}{2} + \frac{1}{Nt}}{2(\exp(\frac{Nt}{2}) - 1)}\,.$$

Similarly, we have:

$$|I_1| < \frac{2\,|f(\varphi_+)| + |f'(\varphi_+)| + \frac{1}{2} + \frac{1}{Nt}}{2(\exp(\frac{Nt}{2}) - 1)}\,.$$

and (1.4.28) follows. □

Remark. If $f(u) = \frac{au^2}{2}$ with some positive constant a, then the Legendre transformation is $g(s) = \frac{s^2}{2a}$ and the functions V_N and u_N which are defined in (1.4.23) and (1.4.22) satisfy the following equations:

$$(V_N)_t = \tfrac{a}{2N}(V_N)_{xx} + \tfrac{1}{2t}V_N \quad \text{and} \quad (u_N)_t + au_N(u_N)_x = \tfrac{a}{2N}(u_N)_{xx}\,,$$

respectively.

Lemma 1.4.7. *Let x_0 be a point of the continuity of the initial state φ. Then*

$$\lim_{t\to 0,\, x\to x_0} u(t,x) = \varphi(x_0)\,. \tag{1.4.35}$$

Proof. First we assume that the function $\varphi(\cdot)$ is continuous in some neighbourhood of the point x_0. If y from this neighbourhood then we have:

$$\tfrac{\partial}{\partial y}\psi(t,x,y) = \varphi(y) - b\left(\tfrac{x-y}{t}\right)\,, \tag{1.4.36}$$

From (1.4.10) it follows that

$$y_0(t,x) \in [x - t\cdot f'(\varphi_+), x - t\cdot f'(\varphi_-)] = [A_{t,x}, B_{t,x}]. \tag{1.4.37}$$

Hence, if (t,x) is close to $(0,x_0)$ then y_0 belongs to a neighbourhood of x_0 where the initial state is continuous. Using (1.4.36), we obtain

$$b\left(\tfrac{x-y_0(t,x)}{t}\right) = \varphi\big(y_0(t,x)\big)\,.$$

It follows from (1.4.37), that $\lim\limits_{t\to 0,\, x\to x_0} y_0(t,x_0) = x_0$ and (1.4.35) follows.

In the general case due to (1.4.12) we have

$$\min_{[A_{t,x},B_{t,x}]}\varphi \leqslant b\left(\tfrac{x-y_0(t,x)}{t}\right) \leqslant \max_{[A_{t,x},B_{t,x}]}\varphi\,. \tag{1.4.38}$$

Since both the r.h.s. and l.h.s. tends to $\varphi(x_0)$ as $(t,x)\to(0,x_0)$, we obtain (1.4.35).

□

Theorem 1.4.8. *Denote*

$$U(t,x) = \min_{y\in\mathbb{R}}(\psi_{t,x}(y)) = \min_{y\in\mathbb{R}}\Big(\Phi(y) + t\cdot g\big(\tfrac{x-y}{t}\big)\Big)\,. \tag{1.4.39}$$

(we recall that the function ψ was defined in (1.4.5)). Then the function $u(t,x)$, defined in (1.4.13) satisfies

$$\int_{x_1}^{x_2} u(t,x)dx = U(t,x_2) - U(t,x_1) \tag{1.4.40}$$

for any real $t > 0$, x_1, x_2; and

$$\int_{t_1}^{t_2} f(u(t,x))dt = U(t_1,x) - U(t_2,x) \tag{1.4.41}$$

For any x and any $t_1 > 0$ and $t_2 > 0$.

Proof. Consider the function $U_N(t,x)$ (cf. (1.4.24))

$$U_N(t,x) = \frac{-1}{N}\log V_N = -\log(V_N^{\frac{1}{N}})\,,$$

where V_N is defined in (1.4.23). For each (t,x) we have:

$$U_N(t,x) = -\log\left(\int_{-\infty}^{+\infty}\Big(\exp\big(-\psi_{t,x}(y)\big)\Big)^N dy\right)^{\frac{1}{N}} \xrightarrow[N\to\infty]{} -\log\max_{y\in\mathbb{R}}\exp\big(-\psi_{t,x}(y)\big) = -\max_{y\in\mathbb{R}}\big(-\psi_{t,x}(y)\big) = \min_{y\in\mathbb{R}}\big(\psi_{t,x}(y)\big).$$

From (1.4.24) we have

$$\int_{x_1}^{x_2} u_N(t,x)dx = U_N(t,x_2) - U_N(t,x_1)\,.$$

By Lebesgue's bounded convergence theorem we can pass to the limit this equality as $N\to\infty$ and get (1.4.40). Similarly from (1.4.26) we have

$$\int_{t_1}^{t_2} f_N(t,x)dt = U_N(t_1,x) - U_N(t_2,x)\,.$$

and (1.4.41) also follows. □

Lemma 1.4.9. *Consider the function $U(t,x)$ defined in (1.4.39) as a family of functions of x, depending on parameter t. Then this family uniformly tends to Φ as $t\to 0$*

$$U(t,x) \rightrightarrows \Phi(x) \quad as \quad t\to 0. \tag{1.4.42}$$

Proof. From the definition of the function U (1.4.39) we have

$$U(t,x) \leqslant \Phi(x) + t\cdot g(0). \tag{1.4.43}$$

Inclusion (1.4.9) states that the minimum in (1.4.39) is achieved on the interval $[A_{t,x}, B_{t,x}]$. Hence we have

$$U(t,x) \geqslant \min_{[A_{t,x},B_{t,x}]}\Phi(\cdot) + t\min g(\cdot). \tag{1.4.44}$$

From (1.4.4) and (1.4.8) it follows that

$$\min_{[A_{t,x},B_{t,x}]}\Phi(\cdot) \geqslant \Phi(x) - t\max\{0, \varphi_- f'(\varphi_-), \varphi_+ f'(\varphi_+)\}. \tag{1.4.45}$$

Using (1.4.43), (1.4.44) and (1.4.45) we arrive at (1.4.42). □

Corollary 1.4.10. *For any C^1-smooth function $q(x)$ with bounded support we have*

$$\lim_{t\to 0}\int_{-\infty}^{+\infty} u(t,x)q(x)dx = -\lim_{t\to 0}\int_{-\infty}^{+\infty} U(t,x)q'_x(x)dx = -\int_{-\infty}^{+\infty}\Phi(x)q'_x(x)dx = \int_{-\infty}^{+\infty}\varphi(x)q(x)dx. \tag{1.4.46}$$

Theorem 1.4.11. *The function $u(t,x)$ defined in (1.4.13) is a weak solution of the single conservation law in sense of definition (1.1.1).*

Proof. The theorem follows from Lemma 1.2.1, Theorem 1.4.6 and corollary 1.4.10. □

Lemma 1.4.12. *Suppose that for some positive K, the initial state φ satisfies the condition*

$$\frac{\varphi(x_2)-\varphi(x_1)}{x_2-x_1} \leqslant K\,, \tag{1.4.47}$$

for all $x_1\neq x_2\in\mathbb{R}$. Then the function (1.4.13) satisfies

$$\frac{u(t,x_2)-u(t,x_1)}{x_2-x_1} \leqslant K\,, \tag{1.4.48}$$

for all $t>0$ and a.a. $x_1\neq x_2\in\mathbb{R}$.

Proof. From (1.4.47) it follows that the function $\varphi(\cdot)$ is continuous everywhere except a countable set, because the function $\varphi(x)-Kx$ is a monotonic (decreasing) function.

If y_0 is (any) minimum point for the function $\psi_{t,x}(y)$, then under the assumption (1.4.47) we have that y_0 is the point of the continuity of the function φ.

Indeed, if y_0 is a point of discontinuity for the function φ, then due to (1.4.47) there exist $\delta>0$ and real numbers $k_l>k_r$ such that

$$\inf_{(y_0-\delta,y_0)}\varphi(y) > k_l > k_r > \sup_{(y_0,y_0+\delta)}\varphi(y)\,.$$

This implies that for $\triangle y\in(0,\delta)$, we have

$$\Phi(y_0+\triangle y) < \Phi(y_0) + k_r\triangle y\,, \qquad \Phi(y_0-\triangle y) < \Phi(y_0) - k_l\triangle y\,.$$

Hence the function $\psi_{t,x}(y)$ can not achieve minimum at the point y_0. This contradiction implies that y_0 is the point of continuity of the initial state $\varphi(\cdot)$.

Let (t, x_1) and (t, x_2) be two points where value y_0 is uniquely defined. Then the initial state $\varphi(\cdot)$ is continuous at the points $y_0(t, x_1)$ and $y_0(t, x_2)$. Hence the function ψ is differentiable w.r.t. y at these points. Using the fact that the derivative at the minimum point (if exists) is equal to zero, we obtain

$$\varphi(y_0(t, x_1)) = b\left(\tfrac{x_1 - y_0(t,x_1)}{t}\right), \qquad \varphi(y_0(t, x_2)) = b\left(\tfrac{x_2 - y_0(t,x_2)}{t}\right).$$

Now we assume that $x_1 < x_2$. If $u(t, x_1) \geqslant u(t, x_2)$ then the l.h.s. in (1.4.48) is less or equal to zero and lemma is proven. Suppose, that $u(t, x_1) < u(t, x_2)$; then since b is an increasing function, we obtain

$$\frac{x_1 - y_0(t, x_1)}{t} < \frac{x_2 - y_0(t, x_2)}{t},$$

or, equivalently, $\quad y_0(t, x_2) - y_0(t, x_1) < x_2 - x_1$. Using this inequality, we obtain:

$$\frac{u(t, x_2) - u(t, x_1)}{x_2 - x_1} = \frac{\varphi(y_0(t, x_2)) - \varphi(y_0(t, x_1))}{x_2 - x_1} < \frac{\varphi(y_0(t, x_2)) - \varphi(y_0(t, x_1))}{y_0(t, x_2) - y_0(t, x_1)} \leqslant K.$$

We arrive at (1.4.48). □

Lemma 1.4.13. *Suppose that there are two real numbers $A_0 < B_0$ such that for some positive K, the initial state φ satisfies the condition*

$$\frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \leqslant K, \tag{1.4.49}$$

for all $x_1 \neq x_2 \in [A_0, B_0]$. Then the function (1.4.13) satisfies

$$\frac{u(t, x_2) - u(t, x_1)}{x_2 - x_1} \leqslant K, \tag{1.4.50}$$

for almost every pair of points (t, x_2), (t, x_1) from the triangle

$$A_0 + tf'(\varphi_+) \leqslant x \leqslant B_0 + tf'(\varphi_-). \tag{1.4.51}$$

Proof. Consider new initial state $\tilde{\varphi}$:

$$\tilde{\varphi}(x) = \begin{cases} \varphi(A_0) & \text{for } x \leqslant A_0 \\ \varphi(x) & \text{for } x \in (A_0, B_0) \\ \varphi(B_0) & \text{for } x \geqslant B_0 \end{cases} \tag{1.4.52}$$

and the correspondent solution $\tilde{u}$ in the sense of explicit formula (1.4.13). For new solution we can apply the previous Lemma and get (1.4.50) for almost all pairs of points with $t > 0$. The proof concludes with the fact that $u = \tilde{u}$ on the triangle (1.4.51). □

Lemma 1.4.14. *Suppose that the function f is uniformly convex:*

$$\frac{f'(u_2) - f'(u_1)}{u_2 - u_1} \geqslant \varepsilon > 0 \qquad \text{for all } u_1 \neq u_2. \tag{1.4.53}$$

(this is a generalisation of condition (1.3.2)). Then the function (1.4.13) satisfies

$$\frac{u(t, x_2) - u(t, x_1)}{x_2 - x_1} \leqslant \frac{1}{\varepsilon t} \qquad \text{for all } t > 0 \text{ and } x_1 \neq x_2 \in \mathbb{R} \setminus D_t, \tag{1.4.54}$$

where for each t $\quad D_t$ is not more than countable set in $\mathbb{R}$.

Proof. The function (1.4.13) satisfies the inequality:

$$\frac{f'\big(u(t, x + {}_\triangle x)\big) - f'\big(u(t, x)\big)}{{}_\triangle x} = \frac{{}_\triangle x - y_0(t, x + {}_\triangle x) + y_0(t, x)}{t\,{}_\triangle x} \leqslant \frac{1}{t}.$$

In the last inequality we have used Lemma 1.4.2 (y_0 is increasing with x). Multiplying this by the inequality

$$\frac{u(t, x + {}_\triangle x) - u(t, x)}{f'\big(u(t, x + {}_\triangle x)\big) - f'\big(u(t, x)\big)} \leqslant \frac{1}{\varepsilon},$$

we arrive at (1.4.54). □

With the help of the explicit construction for the solution we state three lemmas about decay of this solution in the periodic case.

Lemma 1.4.15. *Let the initial state $\varphi(\cdot)$ be a ℓ-periodical function $\big(\varphi(x \pm \ell) = \varphi(x)\big)$ and let M be the mean value of the initial state $\big($i.e., $M = \frac{1}{\ell}\int_0^\ell \varphi(x)dx\big)$, then*

$$\big|f'(u(t, x)) - f'(M)\big| \leqslant \frac{\ell}{t}. \tag{1.4.55}$$

Proof. We can represent the first term in r.h.s. of (1.4.5) as $\int_0^y \varphi(\eta)d\eta = \tilde{\Phi}(y) + My$, where $\tilde{\Phi}(\cdot)$ is an ℓ-periodical function. So we can rewrite (1.4.5) as

$$\psi_{t,x}(y) = \tilde{\Phi}(y) + My + t \cdot g\left(\frac{x - y}{t}\right) = \tilde{\Phi}(y) + \tilde{g}(y).$$

Let y_1 be the point where the function $\tilde{g}(\cdot)$ assume its minimum, then $\frac{\partial \tilde{g}}{\partial y}(y_1) = 0$ or $b\left(\frac{x-y_1}{t}\right) = M$. We note that the function $\tilde{g}(y)$ is decreasing for $y < y_1$, and increasing for $y > y_1$, since this function is convex. The function $\tilde{\Phi}(\cdot)$ is an ℓ-periodical function, so $|y_0 - y_1| \leqslant \ell$. Using that the functions $b(\cdot)$ and $f'(\cdot)$ are inverses of one another, we have:

$$f'(u(t,x)) - f'(M) = f'(b(\tfrac{x-y_0}{t})) - f'(b(\tfrac{x-y_1}{t})) = \tfrac{y_1-y_0}{t}\,,$$

and (1.4.55) follows. □

Lemma 1.4.16. *Suppose that the function f is uniformly convex in sense (1.4.53). If the initial state $\varphi(\cdot)$ is a ℓ-periodical function $\big(\varphi(x \pm \ell) = \varphi(x)\big)$ and M is mean value $\big(M = \frac{1}{\ell}\int_0^\ell \varphi(x)dx\big)$ then*

$$|u(t,x) - M| \leqslant \frac{\ell}{\varepsilon t}\,. \tag{1.4.56}$$

Proof. From (1.4.53) and (1.4.55) we have

$$|u(t,x) - M| \leqslant \frac{|f'(u(t,x)) - f'(M)|}{\varepsilon} \leqslant \frac{\ell}{\varepsilon t}$$

as (1.4.56) states. □

Lemma 1.4.17. *Suppose that the initial state $\varphi(\cdot)$ is ℓ-periodic $\big(\varphi(x \pm \ell) = \varphi(x)\big)$ and M is mean value $\big(M = \frac{1}{\ell}\int_0^\ell \varphi(x)dx\big)$. For any positive c we denote*

$$t_c = \begin{cases} \frac{\ell}{\min\left\{f'(M+c)-f'(M), f'(M)-f'(M-c)\right\}} & \text{if } \varphi_- \leqslant M-c \leqslant M \leqslant M+c \leqslant \varphi_+, \\ \frac{\ell}{f'(M)-f'(M-c)} & \text{if } \varphi_- \leqslant M-c \leqslant M \leqslant \varphi_+ \leqslant M+c, \\ \frac{\ell}{f'(M+c)-f'(M)} & \text{if } M-c \leqslant \varphi_- \leqslant M \leqslant M+c \leqslant \varphi_+, \\ 0 & \text{if } M-c \leqslant \varphi_- \leqslant M \leqslant \varphi_+ \leqslant M+c. \end{cases} \tag{1.4.57}$$

Then for any $t \geqslant t_c$ we have

$$\big|u(t,x) - M\big| \leqslant c\,. \tag{1.4.58}$$

Proof. Applying Lemma 1.4.15 we obtain

$$|f'(u(t,x)) - f'(M)| \leqslant \frac{\ell}{t} \leqslant \frac{\ell}{t_c}.$$

If $\varphi_- \leqslant M-c \leqslant M \leqslant M+c \leqslant \varphi_+$ then we have

$$|f'(u(t,x)) - f'(M)| \leqslant \min\big\{f'(M+c) - f'(M), f'(M) - f'(M-c)\big\}.$$

From this we get the following two inequalities

$$f'(u(t,x)) - f'(M) \leqslant f'(M+c) - f'(M) \tag{1.4.59}$$

and

$$f'(M) - f'(u(t,x)) \leqslant f'(M) - f'(M-c) \tag{1.4.60}$$

Monotonicity of function f' with (1.4.59) implies

$$u(t,x) \leqslant M + c \tag{1.4.61}$$

and with (1.4.60) implies

$$u(t,x) \geqslant M - c\,. \tag{1.4.62}$$

We have proved (1.4.58) for the first case.

If $\varphi_- \leqslant M-c \leqslant M \leqslant \varphi_+ \leqslant M+c$ then we have (1.4.60) for $t > t_c$ and therefore (1.4.62). Inequality (1.4.61) follows from the inequality $u(t,x) \overset{(1.4.10)}{\leqslant} \varphi_+ \leqslant M+c$. The case $M-c \leqslant \varphi_- \leqslant M \leqslant M+c \leqslant \varphi_+$ is considered similarly. In the last case inequality (1.4.58) follows from the inequality $\varphi_- \leqslant u(t,x) \leqslant \varphi_+$ (see (1.4.10)). □

Further on we shall use the following inequality

$$t_c \leqslant \frac{\ell}{\inf\limits_{\substack{x,y\in[\varphi_-,\varphi_+] \\ |x-y|=c}} |f'(x) - f'(y)|}\,. \tag{1.4.63}$$

If f is uniformly convex in the sense (1.4.53), then $|f'(x) - f'(y)| \geqslant \varepsilon|x-y|$ and we have

$$t_c \leqslant \frac{\ell}{\varepsilon c}\,. \tag{1.4.64}$$

As it was advertised in the previous section we prove the following lemma:

Lemma 1.4.18. *The Cauchy problem (1.3.6) has no weak entropy solutions.*

Proof. Suppose that u is a weak entropy solution of the Cauchy problem (1.3.6). Let $u_0(t,x) = \sqrt[3]{\frac{x}{1+t}}$. Then u_0 is the solution of the Cauchy problem (1.3.6) in the sense of formula (1.4.13). Indeed, in this case $b(s) = \sqrt[3]{s}$, $\psi_{t,x}(y) = \frac{3}{4}y^{4/3} + \frac{3}{4}t\big(\frac{x-y}{t}\big)^{4/3}$. Hence $y_{min} = \frac{x}{1+t}$ and $u_0(t,x) = b\big(\frac{x-y_{min}}{t}\big)$.

We show that $u = u_0$ almost everywhere. This leads to contradiction since u_0 is not a entropy solution of the Cauchy problem (1.3.6).

Due to Lemma 1.2.3 there exist a locally Lipschitz function $U(t,x)$ (stream function) defined for $t \geqslant 0$ such that:

$$U_x = u \qquad \text{almost everywhere,} \tag{1.4.65}$$

$$U_t = -f(u) \qquad \text{almost everywhere,}$$

$$U(0,0) = 0.$$

We will show that

$$U = U_0 \tag{1.4.66}$$

where $U_0(t,x) = \min_y\{\frac{3}{4}y^{4/3} + \frac{3}{4}t\big(\frac{x-y}{t}\big)^{4/3}\}$ is the stream function for u_0. We note that (1.4.66) implies that $\frac{\partial}{\partial x}U = \frac{\partial}{\partial x}U_0$ almost everywhere.

Since (1.4.65) holds at least almost everywhere, there exist a sequence $t_k \to 0$ such that for each t_k the equality

$$U_x(t_k,x) = u(t_k,x)$$

holds for almost all x.

We define a function $\tilde{u}_k$ on $\{t \geqslant t_k\} \times \mathbb{R}$ to be the (Lax-Oleinik) solution of the Cauchy problem

$$\begin{cases} \tilde{u}_t + \big(\frac{1}{4}\tilde{u}^4\big)_x = 0, \\ \tilde{u}(t_k,\cdot) = u(t_k,\cdot). \end{cases} \tag{1.4.67}$$

Since the initial state $u(t_k,\cdot)$ satisfies condition (1.4.49) we derive from Lemma 1.4.13 that $\tilde{u}_k$ is an entropy weak solution of the Cauchy problem (1.4.67). Due to choice of t_k, from Lemma 1.2.4 it follows that $u(t,x)$ is also a weak solution of the Cauchy

problem (1.4.67) and therefore an entropy weak solution. Hence by Theorem 1.3.1 we have

$$u(t,x) = \tilde{u}_k(t,x) \qquad \text{almost averywhere in} \qquad \{t \geqslant t_k\} \times \mathbb{R}.$$

Hence for any t_k we have

$$U(t,x) - U(t_k,0) = \min_y \left\{ U(t_k,y) - U(t_k,0) + \tfrac{3}{4}(t-t_k)\left(\tfrac{x-y}{t-t_k}\right)^{4/3} \right\}.$$

Due to Lemma 1.2.4 we have:

$$U(t_k,x) \rightrightarrows U(0,x) \quad \text{as} \quad t_k \to 0.$$

Hence

$$U(t,x) = \min_y \left\{ y^{4/3} + \tfrac{3}{4}t\left(\tfrac{x-y}{t}\right)^{4/3} \right\},$$

and we arrive at (1.4.66). □

1.5 Inviscid limit

In this section we consider the viscous perturbation of the single conservation law (1.1.1) under the ℓ-periodic boundary condition, i.e., we assume that the initial state and, therefore, the solution are ℓ-periodic in x. We study the convergence of solutions as the viscosity tends to zero. We proof that the limit exists and is a weak solution of the single conservation law (1.1.1). We also show that the rate of the convergence has the order of $\sqrt{\nu}$ in the space L_1, where ν is the parameter the perturbation (viscosity).

It is important that if the function f in (1.1.1) is convex, then the resulting weak solution satisfies the entropy condition. By the Oleinik Theorem (see Theorem 1.3.1, see also [26]) this condition implies uniqueness of the weak solution. If the function f is strictly convex (this condition guarantees the existence of the so called Oleinik-Lax solution) then the inviscid limit coincides with the Oleinik-Lax solution, constructed in Section 1.4.

The proof of the convergence is based on Kuznetsov methods. In the article [19] he establish $\sqrt{\nu}$-rate of convergence in the space $L_1(\mathbb{R})$, for the case when the initial

state has bounded variation on $\mathbb{R}$. However his article does not cover the periodic case and this is the subject of this section. Also the constant $\frac{4}{\sqrt{3}}$ occurring in Theorem 1.5.2 is better that the constant $\sqrt{8}$ that would be obtained if we applied Kuznetsov's methods directly.

In this section, when we write $\int ...dx$ (without specifying the limits of integration) we mean $\int_0^\ell ...dx = \int_{x_0}^{x_0+\ell} ...dx$, i.e., the integral over the period or, equivalently, the integral over the circle $S = {}^{\mathbb{R}}\!/_{\ell\mathbb{Z}}$.

Let the function $u = u^\nu(t,x)$ satisfy the equation

$$\frac{\partial}{\partial t}u + \frac{\partial f(u)}{\partial x} = \nu u_{xx} \tag{1.5.1}$$

in the cylinder $[0,+\infty)\times S$, with the initial state

$$u^\nu(0,x) = u_0(x) \in C^\infty(S)\ . \tag{1.5.2}$$

Here $f:\mathbb{R}\to\mathbb{R}$ is an arbitrary C^∞-smooth function (not necessarily convex), $\nu>0$. Hence (see [20]) the function u^ν remains C^∞-smooth for all $t>0$.

We recall that if a function $v(t,x)$ satisfies the equation

$$v_t + \psi(t,x)v_x = \nu v_{xx}\ ,$$

where $\psi(t,x)$ is C^∞-smooth then a maximum principle holds, i.e., for any $t_1<t_2$ we have

$$\inf v(t_1,\cdot) \leqslant \inf v(t_2,\cdot) \leqslant \sup v(t_2,\cdot) \leqslant \sup v(t_1,\cdot)\ .$$

Let the symbol $\dot u$ denote either the function $\frac{\partial u}{\partial t}$ or the function $\frac{\partial u}{\partial x}$ (the following lemma holds for both these cases). We consider the L_1-norm of the function $\dot u$ over the circle S as a function of t, $|\dot u|_{L_1}(t) = \int |\dot u(t,x)|dx$.

Lemma 1.5.1. *The function $t\mapsto |\dot u|_{L_1}(t)$ is non-increasing.*

Proof. The function $\dot u(t,x)$ satisfies the equation

$$\dot u_t + (f'(u)\dot u)_x = \nu \dot u_{xx}\ .$$

We multiply this by a function $\varphi(t,x)\in[T_1,T_2]\times S$, where $T_1<T_2$ and integrate by parts:

$$\int \dot u\varphi\Big|_{t=T_2}dx = \int \dot u\varphi\Big|_{t=T_1}dx + \int_{T_1}^{T_2}\int \dot u(\varphi_t + f'(u)\varphi_x + \nu\varphi_{xx})dxdt\ . \tag{1.5.3}$$

We consider the Cauchy problem for the equation

$$\varphi_t + f'(u)\varphi_x + \nu\varphi_{xx} = 0$$

in the cylinder $[T_1,T_2]\times S$. An initial state should be given at the *top* of the cylinder. We put

$$\varphi(T_2,x) = \operatorname{sign}^\varepsilon(\dot u(T_2,x))\ ,$$

where the function $\operatorname{sign}^\varepsilon(\dot u(T_2,\cdot))$ is a C^∞-smooth approximation of the function $\operatorname{sign}(\dot u(T_2,\cdot))$ such that $|\operatorname{sign}^\varepsilon(\dot u(T_2,x))|\leqslant 1\ \ \forall x$ and distance between these two function in the $L_1(S)$-norm less then ε. Now, by virtue of the maximum principle, we have that $|\varphi(t,x)|\leqslant 1$ for any $(t,x)\in[T_1,T_2]\times S$. Substituting $\varphi(t,x)$ into (1.5.3) we have

$$\int \dot u(T_2,x)\operatorname{sign}^\varepsilon(\dot u(T_2,x))dx = \int \dot u\varphi\Big|_{t=T_1}dx \leqslant \int |\dot u(T_1,x)|dx\ .$$

Letting ε tend to 0, we have

$$\int |\dot u|\Big|_{t=T_2}dx \leqslant \int |\dot u|\Big|_{t=T_1}dx\ . \qquad \square$$

It follows from this lemma that for any $t\geqslant 0$ and $\nu>0$ we have

$$\int \left|\frac{\partial u^\nu}{\partial x}(t,x)\right|dx \leqslant \int \left|\frac{du_0}{dx}\right|dx\ . \tag{1.5.4}$$

For $\nu\in(0,1)$ we have

$$\int |u^\nu(t_2,x) - u^\nu(t_1,x)|\,dx \leqslant C|t_2-t_1|\ , \tag{1.5.5}$$

where the constant C does not depend on ν. We can put

$$C = \int \left|\frac{df(u_0)}{dx}\right|dx + \int \left|\frac{d^2u_0}{dx^2}\right|dx\ .$$

We now investigate the convergence of the solution of the problem (1.5.1), (1.5.2) as $\nu\to 0$.

Theorem 1.5.2. *For any $T \geqslant 0$ the family of functions $u^\nu(T,\cdot)$ converges in the L_1-space to a function $u^0(T,\cdot)$ as $\nu \to 0$ and for any $\nu \geqslant 0$ we have*

$$\int |u^\nu(T,x) - u^0(T,x)|dx \leqslant \frac{4}{\sqrt{3}}\sqrt{T\nu}\int \left|\frac{du_0}{dx}(x)\right| dx\,. \tag{1.5.6}$$

This function u^0 is a weak solution of the Cauchy problem (1.5.1), (1.5.2) with $\nu = 0$.

In what follows we omit the index ν and we denote the limiting function u^0 by v.

Lemma 1.5.3. *For any real number a and for any C^2-smooth, nonnegative function $\psi(t,x)$ we have*

$$\int |u(T,x)-a|\psi(T,x)dx \leqslant \int |u(0,x)-a|\psi(0,x)dx + \\ + \int_0^T\int F(u(t,x),a)\psi_x(t,x) + |u(t,x)-a|(\psi_t(t,x)+\nu\psi_{xx}(t,x))dxdt\,, \tag{1.5.7}$$

where $F(u,a) = \operatorname{sign}(u-a)(f(u)-f(a))$.

Proof. We multiply equation (1.5.1) by a function $\Phi_n'(u)\psi(t,x)$, where $\Phi_n(\cdot)$ is C^2-smooth, convex ($\Phi_n''(\cdot) \geqslant 0$) functions. Then we integrate over the cylinder $[0,T]\times S$:

$$\int_0^T\int \Phi_n'(u)u_t\psi dxdt + \int_0^T\int \Phi_n'(u)f'(u)u_x\psi dxdt = \nu\int_0^T\int u_{xx}\Phi_n'(u)\psi dxdt\,.$$

Noting that $\Phi_n'(u)f'(u)u_x = \frac{\partial}{\partial x}\int_0^u \Phi_n'(\xi)f'(\xi)d\xi$ and integrating by parts, we have:

$$\int \Phi_n(u)\psi\big|_{t=T}dx - \int \Phi_n(u)\psi\big|_{t=0}dx - \\ - \int_0^T\int \Phi_n(u)\psi_t dxdt - \int_0^T\int\left(\int_0^u \Phi_n'(\xi)f'(\xi)d\xi\right)\psi_x dxdt = \\ = -\nu\int_0^T\int u_x^2\Phi_n''(u)\psi dxdt - \nu\int_0^T\int \Phi_n'(u)u_x\psi_x dxdt\,,$$

or

$$\int \Phi_n(u)\psi\big|_{t=T}dx \leqslant \int \Phi_n(u)\psi\big|_{t=0}dx + \int_0^T\int\left(\int_0^u \Phi_n'(\xi)f'(\xi)d\xi\right)\psi_x dxdt + \\ + \int_0^T\int \Phi_n(u)(\psi_t+\nu\psi_{xx})dxdt\,.$$

We now suppose that the $\Phi_n(\cdot)$ approximate the function $|\cdot - a|$ in the supremum norm, such that

$$\frac{d\Phi_n(u)}{du} \to \frac{d}{du}|u-a| = \begin{cases} 1 & \text{if } u > a \\ -1 & \text{if } u < a \end{cases}$$

in $L_1(u_-,u_+)$, where $u_- = \min u_0$, $u_+ = \max u_0$. Thus we can pass to the limit and obtain the inequality (1.5.7). □

It follows from the maximum principle for u^ν and inequality (1.5.4) that the family of the functions

$$\{u^\nu(t,\cdot)\}_{\nu\in(0,1);\,t\in[0,T]} \tag{$\star$}$$

is uniformly bounded and has uniformly bounded variation. By Helly's selection principle (see [15]) any infinite subset of this family contains a pointwise convergent sequence. By Lebesgue's bounded convergence theorem this sequence converges in $L_1(S)$. This means that the family ($\star$) is precompact in $L_1(S)$. From this, inequality (1.5.5) and Ascoli-Arzelà theorem it follows that the family of the functions $\{u^\nu(t,x)\}_{\nu\in(0,1)}$ is precompact in $C\big([0,T];L_1(S)\big)$. Thus there exists an element $v \in C\big([0,T];L_1(S)\big)$ and a sequence $\nu_k \xrightarrow[k\to\infty]{} 0$ such that $u^{\nu_k} \to v$ in $C\big([0,T];L_1(S)\big)$.

We note that this convergence and the maximum principle for u^{ν_k} imply that for any t we have:

$$u_- \leqslant \operatorname{ess\,inf}\, v(t,\cdot) \leqslant \operatorname{ess\,sup}\, v(t,\cdot) \leqslant u_+\,,$$

otherwise $|u^{\nu_k}(t,\cdot) - v(t,\cdot)|_{L_1}$ does not tend to 0. For the next step we need the inequalities

$$\big||u(t,x)-a| - |v(t,x)-a|\big| \leqslant |u(t,x)-v(t,x)|$$

and

$$\big|F(u(t,x),a) - F(v(t,x),a)\big| \leqslant \max_{[u_-,u_+]}|f'(\cdot)||u(t,x)-v(t,x)| \quad \text{for a.a. } (t,x)\,.$$

(The first is just the triangle inequality. To prove the second, one should consider two cases: a) a is between u and v; b) u and v are either both greater than or both less than a. For each t the second inequality holds for almost all x.)

We pass to the limit in inequality (1.5.7) along the sequence $\nu_k \to 0$ (the function $\psi(t,x)$ and the initial state u_0 are fixed):

$$\int |v(T,\tilde{x}) - b|\tilde{\psi}(T,\tilde{x})d\tilde{x} \leqslant \int |v(0,\tilde{x}) - b|\tilde{\psi}(0,\tilde{x})d\tilde{x} + \\ + \int_0^T \int F(v(\tilde{t},\tilde{x}),b)\tilde{\psi}_{\tilde{x}}(\tilde{t},\tilde{x}) + |v(\tilde{t},\tilde{x}) - b|\tilde{\psi}_{\tilde{t}}(\tilde{t},\tilde{x})d\tilde{x}d\tilde{t}\,. \quad (1.5.8)$$

Here we have replaced a with b and t, x, ψ with $\tilde{t}, \tilde{x}, \tilde{\psi}$ for purposes of the remainder of the proof of the theorem.

Lemma 1.5.4. *Suppose that $\Omega : S \to \mathbb{R}$ is an arbitrary nonnegative C^2-smooth function on the circle S; then*

$$\int\int |u(T,x) - v(T,\tilde{x})|\Omega(x-\tilde{x})dxd\tilde{x} \leqslant \int\int |u(0,x) - v(0,\tilde{x})|\Omega(x-\tilde{x})d\tilde{x}dx + \\ + \nu \int_0^T \int\int |u(t,x) - v(t,\tilde{x})|\Omega''(x-\tilde{x})dxd\tilde{x}dt\,.$$

Proof. Suppose $\omega_\varepsilon : \mathbb{R} \to \mathbb{R}$ is a C^2-smooth, even, non-negative function such that $\omega_\varepsilon(z) \equiv 0$ for $|z| \geqslant \varepsilon$ and

$$\int_{-\infty}^{+\infty} \omega_\varepsilon(z)dz = 1\,.$$

We note that for the function $\hat{\psi}(t,\tilde{t},x,\tilde{x}) = \omega_\varepsilon(t-\tilde{t})\Omega(x-\tilde{x})$ we have $\frac{\partial}{\partial t}\hat{\psi} = -\frac{\partial}{\partial \tilde{t}}\hat{\psi}$ and $\frac{\partial}{\partial x}\hat{\psi} = -\frac{\partial}{\partial \tilde{x}}\hat{\psi}$. We substitute $a = v(\tilde{t},\tilde{x})\,,\ \psi(t,x) = \omega_\varepsilon(t-\tilde{t})\Omega(x-\tilde{x})$ into (1.5.7) and integrate with respect to $(\tilde{t},\tilde{x})$ over $[0,T]\times S$. Also, we substitute $b = u(t,x)\,,\ \tilde{\psi}(\tilde{t},\tilde{x}) = \omega_\varepsilon(\tilde{t}-t)\Omega(\tilde{x}-x)$ into inequality (1.5.8) and integrate with respect to (t,x) over $[0,T]\times S$. Then we add the resulting two inequalities.

$$\int_0^T\int\int |u(T,x) - v(\tilde{t},\tilde{x})|\omega_\varepsilon(T-\tilde{t})\Omega(x-\tilde{x})\,dx\,d\tilde{x}\,d\tilde{t} + \\ + \int_0^T\int\int |v(T,\tilde{x}) - u(t,x)|\omega_\varepsilon(T-t)\Omega(x-\tilde{x})\,d\tilde{x}\,dx\,dt \leqslant \\ \leqslant \int_0^T\int\int |u(0,x) - v(\tilde{t},\tilde{x})|\omega_\varepsilon(\tilde{t})\Omega(x-\tilde{x})\,dx\,d\tilde{x}\,d\tilde{t} + \\ + \int_0^T\int\int |v(0,\tilde{x}) - u(t,x)|\omega_\varepsilon(t)\Omega(x-\tilde{x})\,d\tilde{x}\,dx\,dt + \\ + \nu\int_0^T\int_0^T\int\int |u(t,x) - v(\tilde{t},\tilde{x})|\omega_\varepsilon(t-\tilde{t})\tfrac{\partial^2}{\partial x^2}\Omega(x-\tilde{x})\,dx\,d\tilde{x}\,dt\,d\tilde{t}\,.$$

Letting ε tend to 0, we get the statement of the lemma. □

Lemma 1.5.5. *Suppose $\Psi : \mathbb{R} \to \mathbb{R}$ is a nonnegative C^2-smooth function with bounded support; then*

$$\int_{-\infty}^{\infty}\int |u(T,x+y) - v(T,x)|\Psi(y)dxdy \leqslant \int_{-\infty}^{\infty}\int |u(0,x+y) - v(0,x)|\Psi(y)dxdy + \\ + \nu\int_0^T\int_{-\infty}^{\infty}\int |u(t,x+y) - v(t,x)|\Psi''(y)dxdydt\,. \quad (1.5.9)$$

Proof. We apply the previous lemma with $\Omega(\cdot) = \sum_{k=-\infty}^{+\infty}\Psi(\cdot + k\,\ell)$ replacing $x-\tilde{x}$ with y and $\tilde{x}$ with x. □

To get the desired result from (1.5.9) we apply the triangle inequality in the following form:

$$|u(t,x+y) - v(t,x)| = |u(t,x) - v(t,x)| + w(t,x,y)\,,$$

where $|w(t,x,y)| \leqslant |u(t,x+y) - u(t,x)|$. Using equality $u(0,x) \equiv v(0,x)$ and $\int_{-\infty}^{\infty}\Psi''(y)dy = 0$, we have

$$\int |u(T,x) - v(T,x)|dx\int_{-\infty}^{+\infty}\Psi(y)dy \leqslant \int_{-\infty}^{+\infty}\int |u(T,x) - u(T,x+y)|dx\Psi(y)dy + \\ + \int_{-\infty}^{+\infty}\int |u(0,x) - u(0,x+y)|dx\Psi(y)dy + \nu\int_0^T\int_{-\infty}^{+\infty}\int |u(t,x) - u(t,x+y)|dx|\Psi''(y)|dydt. \quad (1.5.10)$$

Further we shall assume that $\int_{-\infty}^{+\infty}\Psi(y)dy = 1$. Using inequalities

$$\int |u(t,x) - u(t,x+y)|dx = \int \left|\int_0^y \frac{\partial u}{\partial x}(t,x+\xi)d\xi\right| dx \leqslant \\ \leqslant \left|\int_0^y \int \left|\frac{\partial u}{\partial x}(t,x+\xi)\right| dxd\xi\right| = |y|\int \left|\frac{\partial u}{\partial x}(t,x)\right| dx \overset{(1.5.4)}{\leqslant} |y|\int \left|\frac{du_0}{dx}\right| dx\,,$$

we rewrite (1.5.10) as follows:

$$\int |u(T,x) - v(T,x)|dx \leqslant \left(2\int_{-\infty}^{+\infty}|y|\Psi(y)dy + T\nu\int_{-\infty}^{+\infty}|y||\Psi''(y)|dy\right)\int \left|\frac{du_0}{dx}\right| dx\,. \quad (1.5.11)$$

Now let the functions $\Psi(y)$ approximate the function $\max\Big(\frac{1}{\varepsilon}\big(1-\frac{|y|}{\varepsilon}\big),0\Big)$. Then the expression in parentheses in (1.5.11) tends to

$$2\int_{-\varepsilon}^{\varepsilon}|y|\tfrac{1}{\varepsilon}\big(1-\tfrac{|y|}{\varepsilon}\big)dy+2\nu T\tfrac{1}{\varepsilon}=\tfrac{2}{3}\left(\varepsilon+\tfrac{3T\nu}{\varepsilon}\right)\ .$$

Minimising with respect to ε, we get estimate (1.5.6).

For any smooth test function $w(t,x)$ vanishing for $|x|+t$ large enough and any $\nu>0$ we have:

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}\big(w_tu^\nu+w_xf(u^\nu)+\nu w_{xx}u^\nu\big)dxdt+\int_{-\infty}^{+\infty}w(0,x)u_0(x)dx=0\,. \quad (1.5.12)$$

Passing this relation to the limit as $\nu\to 0$ we obtain that u^0 satisfies (1.1.3), i.e. u^0 is a weak solution of the Cauchy problem (1.5.1), (1.5.2) with $\nu=0$.

The theorem is proved. □

Under some assumptions (see Lemma 1.5.6) the weak solution v of the limiting equation, constructed in Theorem 1.5.2 (the viscosity weak solution), is an entropy solution.

From this fact and Theorem 1.3.1 it follows that if f is a strictly convex function, then the viscosity solution coincides with the Oleinik-Lax weak solution.

Lemma 1.5.6. *Suppose that $f''\geqslant 0$ and the initial state u_0 satisfies*

$$\frac{\partial u_0}{\partial x}(x)\leqslant K \quad (1.5.13)$$

for all x. Then the weak solution v satisfies

$$\frac{v(t,x_2)-v(t,x_1)}{x_2-x_1}\leqslant K \quad (1.5.14)$$

for any t for almost all x_1, x_2.

Proof. Fix positive ν. Consider the solution u^ν of the Cauchy problem (1.5.1), (1.5.2). Then the function $\dot u=\frac{\partial}{\partial x}u^\nu$ satisfies the following equation:

$$\dot u_t+f'(u^\nu)\dot u_x+f''(u^\nu)(\dot u)^2=\nu\dot u_{xx}$$

and therefore the following inequality holds:

$$\dot u_t+f'(u^\nu)\dot u_x\leqslant\nu\dot u_{xx}.$$

Now by the maximum principal we conclude that

$$\dot u\leqslant K$$

for any t and x or, equivalently,

$$\frac{u^\nu(t,x_2)-u^\nu(t,x_1)}{x_2-x_1}\leqslant K$$

for any $t\geqslant 0$, x_1, x_2. Passing this to the limit as $\nu\to 0$ we arrive at (1.5.14). □

Remark. We use this lemma with $K=\max\limits_{[0,\ell]}\frac{\partial u_0}{\partial x}$.

Chapter 2

One dimensional Burgers Equation

2.1 Introduction

In this Chapter we study the initial value problem for the one dimensional ($\dim x = 1$) parabolic equation of Burgers type:

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}f(u) = \nu u_{xx}\,, \tag{2.1.1}$$

with the initial state

$$u(0,x) = u_0(x)\,, \tag{2.1.2}$$

and satisfying the periodical boundary conditions

$$u(t, x \pm \ell) = u(t,x)\,. \tag{2.1.3}$$

Here ν is a small (< 1) positive parameter, the initial state is a C^∞-smooth and ℓ-periodic function $u_0(\cdot)$. The function $f : \mathbb{R} \to \mathbb{R}$ is C^∞-smooth and strictly convex, i.e., it is convex and the graph of f does not contain any straight interval.

The classical theory of parabolic equations (see [20]) states that problems (2.1.1), (2.1.2), (2.1.3) has a unique solution $u = u^\nu(t,x)$, for every $\nu > 0$ which is a C^∞-smooth. It has also been known that while $\nu \to +0$ the solution of the problem (2.1.1), (2.1.2), (2.1.3) converges to the so called (weak) entropy solution of quasilinear equation

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0\,. \tag{2.1.1$'$}$$

See [16], where this fact has been studied under the most general assumptions. Moreover in [19] it has been shown that rate of convergence in the $L_1(\mathbb{R})$ norm is at least $Const \cdot \sqrt{\nu}$. See Section 1.5 for the proof of this fact for our situation.

We remark that equation (2.1.1$'$) is, in general, not solvable even in the class of continuous functions. For this reason it is normal to consider weak solutions in the sense of corresponding integral identity. However, this approach leads to loss of uniqueness. Imposing an entropy condition ensures uniqueness and the resulting unique solution is the most natural from the physical point of view.

We shall study behaviour of the H^m-Sobolev seminorms

$$\|u\|_m^2 = \int \left|\frac{\partial^m u}{\partial x^m}\right|^2 dx$$

as $\nu \to 0$. In this chapter, we write $\int ...dx$ (omitting the integration limits) for $\int_0^\ell ...dx = \int_{x_0}^{x_0+\ell} ...dx$, i.e., an integral over the period or, equivalently, an integral over the circle $S = {}^{\mathbb{R}}\!/_{\ell\mathbb{Z}}$. Henceforth we will omit the prefix semi- from the word seminorm.

The H^m-Sobolev norm is a function, which depends on t. We consider the average of this function over $[0,T]$:

$$\left\langle \|u\|_m^2 \right\rangle_{[0,T]} = \frac{1}{T}\int_0^T \int \left|\frac{\partial^m u}{\partial x^m}\right|^2 dxdt\,.$$

We shall show that for any $m \geqslant 1$, ν is small enough and $t \in \mathbb{R}_+$ we have that

$$\|u\|_m \leqslant C_m \left(\frac{1}{\nu}\right)^{m-1/2}\,. \tag{2.1.4}$$

On the other hand, there exists a number T, which depends on the initial condition of problem (2.1.1), (2.1.2), (2.1.3) (see formula (2.3.1)) such that

$$\left\langle \|u\|_m^2 \right\rangle_{[0,T]}^{1/2} \geqslant c_m \left(\frac{1}{\nu}\right)^{m-1/2}\,. \tag{2.1.5}$$

Here the constants c_m and C_m does not depend on ν, and could be explicitly expressed in the terms of the C^k-norm ($k \leqslant m+1$) of the initial state of problem (2.1.1), (2.1.2), (2.1.3).

The H^m-Sobolev norm can be expressed in terms of Fourier coefficients:

$$\|u\|_m^2 = \left(\frac{2\pi}{\ell}\right)^{2m} \sum_{s=-\infty}^{+\infty} s^{2m} |\hat{u}_s|^2 \,,$$

where $\hat{u}_s = \frac{1}{\sqrt{\ell}} \int u(x) e^{-\frac{2\pi i s x}{\ell}} dx$. Together with (2.1.4) and (2.1.5) this relation makes it possible to get some information about the behaviour of the Fourier coefficients of the solution as $\nu \to +0$. Let $\theta > 0$ be a fixed number. By $\hat{A}^2_{s,\theta}$ we denote the average of $|\hat{u}_k|^2(t)$ over $t \in [0;T]$ and $\left(\frac{1}{\nu}\right)^{s-\theta} < |k| \leqslant \left(\frac{1}{\nu}\right)^{s+\theta}$, i.e.,

$$\hat{A}^2_{s,\theta} = \frac{1}{T} \int_0^T \frac{\sum\limits_{\nu^{-s+\theta} < |k| \leqslant \nu^{-s-\theta}} |\hat{u}_k|^2(t)}{\sum\limits_{\nu^{-s+\theta} < |k| \leqslant \nu^{-s-\theta}} 1} dt \,,$$

here $s > 0$. Then for all small enough ν the values $\hat{A}^2_{s,\theta}$ satisfy to the following estimates:

$$\hat{A}^2_{s,\theta} < \tfrac{\ell}{\pi^2} \left|\tfrac{du_0}{dx}\right|^2_{L_1} \nu^{2s} \qquad \text{for } s > 0 \quad , \tag{2.1.6}$$

$$\hat{A}^2_{s,\theta} < \left(\tfrac{\ell}{2\pi}\right)^{2m} C_m^2 \, \nu^{2+2m(s-1-\theta)} \qquad \forall m > 0 \qquad \text{for } s > 1+\theta \,, \tag{2.1.7}$$

$$\hat{A}^2_{1,\theta} > \left(\tfrac{\ell}{2\pi}\right)^2 \tfrac{c_1^2}{4} \, \nu^{2+2\theta} \qquad \text{for } s = 1 \quad . \tag{2.1.8}$$

This chapter is devoted to the proving of estimates (2.1.4) — (2.1.8) for the problem (2.1.1), (2.1.2), (2.1.3).

We note that relations (2.1.6) — (2.1.8) are closely related to the so called Kolmogorov asymptotics for freely decaying turbulence (i.e., unforced turbulence), described by Burgers equation. In particularly it follows from (2.1.6) — (2.1.8) that the Kolmogorov dissipation scale of solutions of Burgers equation (2.1.1) is equal to ν and the exponent in the spectral law is equal to two (see [8], Chpt. 6; cf. [18],[6] and Section 3.4 of this thesis).

The following interpolation inequalities will be used subsequently:

$$\|u\|_k \leqslant \|u\|_0^{1-\frac{k}{m}} \|u\|_m^{\frac{k}{m}} \qquad (0 \leqslant k \leqslant m) \,, \tag{2.1.9}$$

$$|u|_k \leqslant C_{mk} |u|_0^{1-\frac{k}{m}} |u|_m^{\frac{k}{m}} \qquad (0 \leqslant k \leqslant m) \,. \tag{2.1.10}$$

Here $|u|_k = \max_{x \in S} \left|\frac{d^k u}{dx^k}\right|$. The proof of inequality (2.1.9) is simple:

$$\left(\frac{\ell}{2\pi}\right)^{2k} \|u\|_k^2 = \sum \left(|\hat{u}_s|^2\right)^{\frac{m-k}{m}} \cdot \left(|\hat{u}_s|^2 \cdot s^{2m}\right)^{\frac{k}{m}} \leqslant$$

$$\leqslant \left(\sum |\hat{u}_s|^2\right)^{\frac{m-k}{m}} \cdot \left(\sum |\hat{u}_s|^2 \cdot s^{2m}\right)^{\frac{k}{m}} = \|u\|_0^{1-\frac{k}{m}} \cdot \left(\left(\frac{\ell}{(2\pi)}\right)^{2m} \|u\|_m^2\right)^{\frac{k}{m}} \,.$$

The proof of inequality (2.1.10) with the optimal constants is not trivial. It can be found in [14]. The optimal values of the constants C_{mk} are:

$$C_{mk} = \frac{K_{m-k}}{K_m^{1-\frac{k}{m}}} \,, \text{ where} \qquad K_i = \frac{4}{\pi}\left(1 + \frac{(-1)^{i+1}}{3^{i+1}} + \frac{1}{5^{i+1}} + \frac{(-1)^{i+1}}{7^{i+1}} + ...\right) \,.$$

Further on, we shall use the following simple fact:

$$(C_{m1})^m \leqslant 2 \,. \tag{2.1.11}$$

(See Appendix for the proof.) Besides this we shall use the Gagliardo-Nirenberg inequality for the periodic functions:

$$\left|\frac{\partial^\alpha u}{\partial x^\alpha}\right|_{L_{\frac{2m}{\alpha}}} \leqslant 4^{\alpha(m-\alpha)} |u|_{L_\infty}^{1-\frac{\alpha}{m}} \left|\frac{\partial^m u}{\partial x^m}\right|^{\alpha/m}_{L_2} \,, \tag{2.1.12}$$

where α is an integer between 0 and m (see [13], pp. 106-107).

We denote $\frac{\partial^m u}{\partial x^m}$ by $u^{(m)}$, similarly $u_x = \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, and $u_t = \frac{\partial u}{\partial t}$. Unless otherwise stated we let L_p denote the space $L_p(S)$.

2.2 Upper estimates

Let u_+ and u_- be maximum and minimum value of the initial state $u_0 = u_0(x)$. The oscillation of the solution at time t is $\operatorname{osc}\limits_x u(t,x) = \max\limits_x u(t,x) - \min\limits_x u(t,x)$. We define the constant D to be

$$D = \frac{1}{2} \operatorname*{osc}_x u\Big|_{t=0} = \frac{u_+ - u_-}{2} \,. \tag{2.2.1}$$

Then, by virtue of maximum principle, we have $u_- \leqslant u(t,x) \leqslant u_+$ and

$$\frac{1}{2} \operatorname*{osc}_x u(t,x) \leqslant D \quad \text{ for all } t \,. \tag{2.2.2}$$

Multiplying equation (2.1.1) by u and integrating over period we get:

$$\int uu_t dx = -\nu \int u_x^2 dx \,, \tag{2.2.3}$$

or

$$\nu \int u_x^2 dx = -\int uu_t dx \leqslant \frac{\operatorname*{osc}_x u(t,x)}{2} \left|u_t\right|_{L_1} \leqslant D \cdot \left|u_t\right|_{t=0}\Big|_{L_1} .$$

In the last inequality we have used inequality (2.2.2) and the fact that $\left|u_t\right|_{L_1} = \int \left|\frac{\partial u}{\partial t}(t,x)\right| dx$ does not increase with t, see lemma 1.5.1. We denote constant C_1 as

$$C_1 = \sqrt{D \cdot \left(\int \left|\tfrac{\partial f(u_0)}{\partial x}\right| dx + \int \left|\tfrac{\partial^2 u_0}{\partial x^2}\right| dx \right)} \,. \tag{2.2.4}$$

Expressing $u_t\big|_{t=0}$ via equation (2.1.1) we have

$$\|u^\nu\|_1 \leqslant C_1 \nu^{-1/2} \tag{2.2.5}$$

for all $t \geqslant 0$ and $\nu \in (0,1)$.

Differentiating equation (2.1.1) $m-1$ times with respect to x, multiplying by $u^{(m+1)}$, and integrating over period we obtain:

$$\frac{1}{2}\frac{d}{dt}\int \left(\frac{\partial^m u}{\partial x^m}\right)^2 dx = -\nu \int \left(\frac{\partial^{m+1} u}{\partial x^{m+1}}\right)^2 dx + \int \frac{\partial^{m+1} u}{\partial x^{m+1}} \cdot \frac{\partial^m f(u)}{\partial x^m} dx \,. \tag{2.2.6}$$

To prove the main estimates of this section we need the following technical inequality.

Lemma 2.2.1. *For all $m \geqslant 1$ we have:*

$$\int \left|\frac{\partial^{m+1} u}{\partial x^{m+1}}\right| \cdot \left|\frac{\partial^m f(u)}{\partial x^m}\right| dx \leqslant B_m \cdot \|u\|_m \|u\|_{m+1} \,, \tag{2.2.7}$$

where

$$B_m = Q_m |f(\cdot)|_{C^m[u_-,u_+]} \max\left(1, |u_0|_{L_\infty}^{m-1}\right) \,, \tag{2.2.8}$$

and Q_m depends on m only.

Proof. By induction on m it is easy to prove that

$$\frac{\partial^m f(u)}{\partial x^m} = \sum_{k=1}^m \frac{d^k f(u)}{du^k} \left(\sum_{\substack{\alpha_1+\ldots+\alpha_k=m \\ 1\leqslant\alpha_1\leqslant\ldots\leqslant\alpha_k}} A^{k,m}_{\alpha_1,\ldots,\alpha_k} u^{(\alpha_1)} u^{(\alpha_2)} \ldots u^{(\alpha_k)} \right) \,,$$

where the constants $A^{k,m}_{\alpha_1,\ldots,\alpha_k}$ are nonnegative integers. It follows from this that

$$\int \left|\frac{\partial^{m+1} u}{\partial x^{m+1}}\right| \cdot \left|\frac{\partial^m f(u)}{\partial x^m}\right| dx \leqslant$$

$$\leqslant \sum_{k=1}^m \sum_{\substack{\alpha_1+\ldots+\alpha_k=m \\ 1\leqslant\alpha_1\leqslant\ldots\leqslant\alpha_k}} A^{k,m}_{\alpha_1,\ldots,\alpha_k} \left|\frac{d^k f(u)}{du^k}\right| \int \left|u^{(m+1)}\right| \left|u^{(\alpha_1)}\right| \ldots \left|u^{(\alpha_k)}\right| dx \leqslant$$

we use Hölder's inequality (note: $\frac{1}{2} + \frac{\alpha_1}{2m} + \ldots + \frac{\alpha_k}{2m} = 1$):

$$\leqslant \sum_{k=1}^m \sum_{\substack{\alpha_1+\ldots+\alpha_k=m \\ 1\leqslant\alpha_1\leqslant\ldots\leqslant\alpha_k}} A^{k,m}_{\alpha_1,\ldots,\alpha_k} \left|\frac{d^k f(u)}{du^k}\right|_{L_\infty} \left|u^{(m+1)}\right|_{L_2} \left|u^{(\alpha_1)}\right|_{L_{\frac{2m}{\alpha_1}}} \ldots \left|u^{(\alpha_k)}\right|_{L_{\frac{2m}{\alpha_k}}} \leqslant$$

we use inequality (2.1.12), noting that $\prod_{i=1}^k 4^{\alpha_i(m-\alpha_i)} \leqslant 4^{m^2}$:

$$\leqslant \sum_{k=1}^m \sum_{\substack{\alpha_1+\ldots+\alpha_k=m \\ 1\leqslant\alpha_1\leqslant\ldots\leqslant\alpha_k}} A^{k,m}_{\alpha_1,\ldots,\alpha_k} \left|\frac{d^k f(u)}{du^k}\right|_{L_\infty} |u^{(m+1)}|_{L_2} \, 4^{m^2} |u|_{L_\infty}^{k-1} |u^{(m)}|_{L_2} =$$

$$= 4^{m^2} \left(\sum_{k=1}^m \sum_{\substack{\alpha_1+\ldots+\alpha_k=m \\ 1\leqslant\alpha_1\leqslant\ldots\leqslant\alpha_k}} A^{k,m}_{\alpha_1,\ldots,\alpha_k} \left|\frac{d^k f(u)}{du^k}\right|_{L_\infty} |u|_{L_\infty}^{k-1} \right) \|u\|_m \|u\|_{m+1} \,.$$

It follows from the maximum principle that $|u|_{L_\infty} \leqslant |u_0|_{L_\infty}$. Thus the lemma is proved. □

Combining (2.2.6) and (2.2.7) we obtain:

$$\frac{1}{2}\frac{d}{dt}\|u\|_m^2 \leqslant -\nu \|u\|_{m+1}^2 + B_m \|u\|_m \|u\|_{m+1} = \|u\|_{m+1} \left(-\nu \|u\|_{m+1} + B_m \|u\|_m\right). \tag{2.2.9}$$

Theorem 2.2.2. *Suppose $m \in \mathbb{N}$. We define ν'_m by*

$$\nu'_m = B_m \cdot \left(\frac{C_1}{\sqrt{B_m}\, \|u_0\|_m} \right)^{\frac{1}{m-1/2}} \,.$$

If $\nu \leqslant \nu'_m$ then for all $t \geqslant 0$ we have

$$\|u(t,\cdot)\|_m \leqslant B_m^{m-1} C_1 \nu^{-m+\frac{1}{2}} \,. \tag{2.2.10}$$

Here C_1 and B_m are defined in (2.2.4) and (2.2.8) respectively.

If $\nu \geqslant \nu'_m$ then

$$\|u(t,\cdot)\|_m \leqslant \|u(0,\cdot)\|_m \,. \tag{2.2.11}$$

Equivalently, for any $t \geqslant 0$ and $\nu > 0$, we have

$$\|u(t,\cdot)\|_m \leqslant \max\left\{\|u(0,\cdot)\|_m \,,\, B_m^{m-1} C_1 \nu^{-m+\frac{1}{2}}\right\} \tag{2.2.12}$$

Proof. Substituting u_x for u in (2.1.9), we obtain:

$$\|u\|_m \leqslant \|u\|_1^{\frac{1}{m}} \cdot \|u\|_{m+1}^{\frac{m-1}{m}} \,. \tag{2.2.13}$$

We rearrange this as $\|u\|_{m+1} \geqslant \|u\|_m \left(\frac{\|u\|_m}{\|u\|_1}\right)^{\frac{1}{m-1}}$. Combining (2.2.9) and the last inequality we have:

$$\frac{1}{2}\frac{d}{dt}\|u\|_m^2 \leqslant \|u\|_{m+1}\|u\|_m \left(-\nu\left(\frac{\|u\|_m}{\|u\|_1}\right)^{\frac{1}{m-1}} + B_m\right) \,.$$

Hence if $\|u\|_m > \left(\frac{B_m}{\nu}\right)^{m-1}\|u\|_1$ then $\|u\|_m$ is decreasing. Combining this with (2.2.5) we have:

$$\text{if} \quad \|u(t,\cdot)\|_m > B_m^{m-1} C_1 \nu^{-m+\frac{1}{2}} \,, \quad \text{then} \quad \|u\|_m \ \text{is decreasing.} \tag{2.2.14}$$

Let us consider the case $\nu \leqslant \nu'_m$. This inequality means that (2.2.10) holds at $t = 0$. Combining this with (2.2.14) we conclude that $\|u\|_m$ can never surpass the r.h.s. of (2.2.10).

Suppose now that $\nu > \nu'_m$. This means that $\|u(0,\cdot)\|_m > B_m^{m-1} C_1 \nu^{-m+\frac{1}{2}}$, and thus (2.2.11) follows from (2.2.14). □

2.3 Lower estimates

Let $M = \frac{1}{\ell}\int u_0 dx$ denote the mean value of the initial state.

From Lemma 1.4.17 with $c = \frac{|u_0(\cdot)-M|_{L_2}}{2\sqrt{\ell}}$ and inequality (1.4.63) with the same c, it follows that for

$$T = \frac{\ell}{\inf\limits_{\substack{x,y\in[\varphi_-,\varphi_+] \\ |x-y|=\frac{|u_0(\cdot)-M|_{L_2}}{2\sqrt{\ell}}}} |f'(x) - f'(y)|} \tag{2.3.1}$$

we have

$$\left|u^0(T,x) - M\right|_{L_\infty} \leqslant \frac{|u_0(\cdot) - M|_{L_2}}{2\sqrt{\ell}} \,. \tag{2.3.2}$$

It follows from this that

$$\int \left(u^0(T,x) - M\right)^2 dx \leqslant \frac{1}{4}|u_0 - M|_{L_2}^2 \,. \tag{2.3.3}$$

In what follows we assume that $M = 0$, in order to simplify formulae. This assumption has been made without loss of generality because we can make the substitution $u \to u - M$, $f(\cdot) \to f(\cdot + M)$.

Theorem 2.3.1. *Suppose $m \in \mathbb{N}$ and $\nu < \nu_0'' = \frac{1}{3072}\frac{|u_0|_{L_2}^4}{T \cdot |u_0|_{L_\infty}^2 \cdot \left(\int \left|\frac{du_0}{dx}\right| dx\right)^2}$; then we have*

$$\left(\int_0^T \|u(t,\cdot)\|_m^2 dt\right)^{1/2} \geqslant c_m \left(\frac{1}{\nu}\right)^{m-1/2} \,. \tag{2.3.4}$$

Here T is defined in (2.3.1) and c_m does not depend on ν.

Proof. We rearrange (2.2.3) as

$$\nu \int \left(u_x^\nu(t,x)\right)^2 dx = -\frac{1}{2}\frac{d}{dt}\int \left(u^\nu(t,x)\right)^2 dx \,.$$

We integrate this over $[0,T]$ with respect to t

$$\int_0^T \nu\|u^\nu\|_1^2 dt = \frac{1}{2}\int \left(u^\nu(0,x)\right)^2 dx - \frac{1}{2}\int \left(u^\nu(T,x)\right)^2 dx \,. \tag{2.3.5}$$

We shall have proved inequality (2.3.4) for $m = 1$ if we can show that the last term in (2.3.5) is small. Suppose $\nu < \nu_0''$; then using inequality (1.5.6) we have:

$$\begin{aligned}\int \left(u^\nu(T,x)\right)^2 dx - \int \left(u^0(T,x)\right)^2 dx \leqslant \\ \leqslant \int \left|u^\nu(T,x) - u^0(T,x)\right| \cdot \left|u^\nu(T,x) + u^0(T,x)\right| dx \leqslant \\ \leqslant \frac{4}{\sqrt{3}}\sqrt{T\nu} \cdot \int \left|\frac{du_0}{dx}\right| dx \cdot 2\,|u_0|_{L_\infty} \leqslant \frac{1}{12}\,|u_0|_{L_2}^2 \,.\end{aligned}$$

Combining this with (2.3.5) and (2.3.3) we have

$$\int_0^T \nu\|u^\nu\|_1^2 dt \geqslant \frac{1}{2}|u_0|_{L_2}^2 - \frac{1}{2}\left(\frac{1}{4}|u_0|_{L_2}^2 + \frac{1}{12}|u_0|_{L_2}^2\right) = \frac{1}{3}|u_0|_{L_2}^2 \,,$$

or

$$\frac{1}{T}\int_0^T \|u^\nu\|_1^2 dt \geqslant \frac{|u_0|_{L_2}^2}{3T\nu} \,. \tag{2.3.6}$$

So inequality (2.3.4) is proved for $m = 1$ with the constant $c_1 = \frac{|u_0|_{L_2}}{\sqrt{3T}}$.

It follows from (2.2.2) and the interpolation inequality (2.1.10) with $k = 1$ that $|u_x|_{L_\infty} \leqslant C_{m1} \cdot D^{\frac{m-1}{m}} \cdot |u^{(m)}|^{\frac{1}{m}}_{L_\infty}$, where D is defined in (2.2.1). Using (2.1.11), we obtain $\left|u^{(m)}\right|_{L_\infty} \geqslant \frac{\left(|u_x|_{L_\infty}\right)^m}{2D^{m-1}}$. Using inequality (1.5.4) we get $\left|\frac{\partial u^\nu}{\partial x}\right|_{L_\infty} \geqslant \frac{\|u^\nu\|_1^2}{\left|\frac{du_0}{dx}\right|_{L_1}}$. Thus we have

$$\left|u^{(m)}\right|_{L_\infty} \geqslant \frac{\|u^\nu\|_1^{2m}}{2D^{m-1}\left|\frac{du_0}{dx}\right|^m_{L_1}} \,. \tag{2.3.7}$$

Lemma 2.3.2. *Let v be a periodic function that achieves the value 0 then*

$$|v|^2_{L_\infty} \leqslant |v|_{L_2} \cdot |v_x|_{L_2} \,.$$

Proof. For any y we have

$$v^2(y) = \int\limits_{v^{-1}(0)}^{y} \frac{d}{dx} v^2(x)dx \leqslant \frac{1}{2}\int \left|\frac{d}{dx}v^2(x)\right| dx = \int |v(x)||v'(x)|dx \leqslant |v|_{L_2}\,|v_x|_{L_2} \,. \quad \square$$

We apply this lemma with $v = \frac{\partial^m u}{\partial x^m}$ and using inequality (2.2.13) we get

$$\left|u^{(m)}\right|_{L_\infty} \leqslant \|u\|^{1/2}_m \cdot \|u\|^{1/2}_{m+1} \leqslant \|u\|^{\frac{1}{2m}}_1 \cdot \|u\|^{\frac{2m-1}{2m}}_{m+1} \,.$$

Hence

$$\|u\|_{m+1} \geqslant \left(\frac{|u^{(m)}|^{2m}_{L_\infty}}{\|u\|_1}\right)^{\frac{1}{2m-1}} \overset{(2.3.7)}{\geqslant} \frac{\|u\|^{2m+1}_1}{\left(2D^{m-1}\left|\frac{du_0}{dx}\right|^m_{L_1}\right)^{\frac{2m}{2m-1}}} \,.$$

Let's square the left and right hand sides of this inequality. Using the inequality $\frac{1}{T}\int_0^T \|u\|_1^{4m+2}dt \geqslant \left(\frac{1}{T}\int_0^T \|u\|_1^2 dt\right)^{2m+1}$, we take average over $t \in [0,T]$:

$$\frac{1}{T}\int_0^T \|u\|^2_{m+1}dt \geqslant \frac{\left(\frac{1}{T}\int_0^T \|u\|_1^2 dt\right)^{2m+1}}{\left(2D^{m-1}\left|\frac{du_0}{dx}\right|^m_{L_1}\right)^{\frac{4m}{2m-1}}} \overset{(2.3.6)}{\geqslant} \frac{\left(\frac{|u_0|^2_{L_2}}{3T\nu}\right)^{2m+1}}{\left(2D^{m-1}\left|\frac{du_0}{dx}\right|^m_{L_1}\right)^{\frac{4m}{2m-1}}} \,.$$

Thus inequality (2.3.4) is proved for all $m \geqslant 1$. $\square$

It is interesting to compare inequality (2.3.4) for the case $m = 1$ with the upper estimate for the averaged L_1-norm (which follows from inequality (1.5.4)):

$$\left\langle \int u'(x)^2 dx\right\rangle^{1/2} \geqslant c_1\nu^{-1/2} \,, \qquad \left\langle \int |u'(x)|dx\right\rangle \leqslant Const \,.$$

Broadly speaking this means that the graph of the gradient $u'(x)$ is a curve with high and infrequent peaks. This behaviour is closely related with the phenomenon of intermittency from the theory of turbulence, see [8], Chpt. 8.

2.4 Behaviour of the Fourier coefficients

In this section we prove estimates (2.1.6) — (2.1.8).

The Fourier coefficients of a function, defined on the circle S are the coefficients of its expansion in the orthonormal basis $e_k(x) = \frac{1}{\sqrt{\ell}}\exp\left(\frac{2\pi i k x}{\ell}\right)$, $k \in \mathbb{Z}$. The expansion of $u(x)$ in this basis has the form

$$u(x) = \sum_{k=-\infty}^{+\infty} \hat{u}_k e_k(x) \,, \quad \text{where } \hat{u}_k = \frac{1}{\sqrt{\ell}}\int u(x)\exp\left(\frac{-2\pi i k x}{\ell}\right)dx \,.$$

The H^m-Sobolev norm can be written as

$$\|u\|^2_m = \left(\frac{2\pi}{\ell}\right)^{2m}\sum_{k=-\infty}^{+\infty} k^{2m}|\hat{u}_k|^2 \,.$$

Let $\hat{a}_k^2 = \frac{1}{T}\int_0^T |\hat{u}_k(t)|^2 dt$; then we have

$$\frac{1}{T}\int_0^T \|u\|^2_m dt = \left(\frac{2\pi}{\ell}\right)^{2m}\sum_{k=-\infty}^{+\infty} k^{2m}\hat{a}_k^2 \,.$$

Combining this formula with the results of sections 2.2 and 2.3 we see that for any $m \geqslant 1$ and for any ν small enough $\big($namely $\nu < \nu_0 = \min(1, \nu_0', \nu_0'')\big)$, it follows that

$$\tilde{c}_m\nu^{-2m+1} < \sum_{k=-\infty}^{+\infty} k^{2m}\hat{a}_k^2 < \widetilde{C}_m\nu^{-2m+1} \,, \tag{$*$}$$

where $\tilde{c}_m = \left(\frac{\ell}{2\pi}\right)^{2m}c_m^2$ and $\widetilde{C}_m = \left(\frac{\ell}{2\pi}\right)^{2m}C_m^2$. We define the constant V to be $V = \frac{\ell}{4\pi^2}\left|\frac{du_0}{dx}\right|^2_{L_1}$. Then

$$|\hat{u}_k| = \left|\frac{i\sqrt{\ell}}{2\pi k}\int \frac{\partial u}{\partial x}\exp\left(\frac{-2\pi i k x}{\ell}\right)dx\right| \leqslant \frac{\sqrt{\ell}}{2\pi|k|}\left|\frac{\partial u}{\partial x}\right|_{L_1} \leqslant \frac{V^{1/2}}{|k|} \,.$$

In the last estimate we have used inequality (1.5.4). Thus we have

$$\hat{a}_k^2 \leqslant \frac{V}{k^2} \,.$$

Let s be a nonnegative real and θ be a positive real. By $\hat{A}^2_{s,\theta}$ we denote the average of the values $\hat{a}_k^2$ over $|k| \in (\nu^{-s+\theta}; \nu^{-s-\theta}]$. Suppose $\nu^\theta < \frac{1}{2}$. This condition ensures that set of indexes k, over which the averaging takes place is not empty. Thus, using the simple fact that $\frac{1}{k^2} < \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} \frac{dx}{x^2}$, we have

$$\hat{A}^2_{s,\theta} < \frac{V\int_{[\nu^{-s+\theta}]+\frac{1}{2}}^{[\nu^{-s-\theta}]+\frac{1}{2}} \frac{1}{x^2}dx}{[\nu^{-s-\theta}] - [\nu^{-s+\theta}]} = \frac{V}{\left([\nu^{-s-\theta}] + \frac{1}{2}\right) - \left([\nu^{-s+\theta}] + \frac{1}{2}\right)} < 4V\nu^{2s}$$

and estimate (2.1.6) is proved.

Let us turn to estimate (2.1.7). Suppose $1 < s-\theta < s < s+\theta$, $\nu^\theta < \frac{1}{2}$, and $(*)$ holds. Then

$$\sum_{|k|>\nu^{-s+\theta}} \hat{a}_k^2 \leqslant (\nu^{s-\theta})^{2m} \sum_{|k|>\nu^{-s+\theta}} k^{2m}\hat{a}_k^2 \leqslant (\nu^{s-\theta})^{2m} \sum_{k=-\infty}^{+\infty} k^{2m}\hat{a}_k^2 \leqslant \widetilde{C}_m \nu^{2m(s-\theta-1)+1}.$$

Thus we have

$$\hat{A}_{s,\theta}^2 \leqslant \frac{\sum_{|k|>\nu^{-s+\theta}} \hat{a}_k^2}{2\left([\nu^{-s-\theta}] - [\nu^{-s+\theta}]\right)} \leqslant \frac{\widetilde{C}_m \nu^{2m(s-\theta-1)+1}}{\nu^{-s-\theta}} < \widetilde{C}_m \nu^{2+2m(s-\theta-1)}$$

and estimate (2.1.7) is proved.

Now let us turn to (2.1.8). Suppose $\frac{2V+\widetilde{C}_2}{\tilde{c}_1}\nu^\theta < \frac{1}{2}$ and $(*)$ holds. We have

$$\sum_{|k|>\nu^{-1-\frac{\theta}{2}}} k^2\hat{a}_k^2 \leqslant \nu^{2+\theta} \sum_{|k|>\nu^{-1-\frac{\theta}{2}}} k^4\hat{a}_k^2 \leqslant \widetilde{C}_2 \nu^{-1+\theta}$$

and

$$\sum_{|k|\leqslant\nu^{-1+\theta}} k^2\hat{a}_k^2 \leqslant 2 \sum_{0<k\leqslant\nu^{-1+\theta}} V \leqslant 2V\nu^{-1+\theta}\,.$$

Therefore we obtain

$$\sum_{\nu^{-1+\theta}<|k|\leqslant\nu^{-1-\theta}} \hat{a}_k^2 \geqslant \nu^{2+\theta} \sum_{\nu^{-1+\theta}<|k|\leqslant\nu^{-1-\frac{\theta}{2}}} k^2\hat{a}_k^2 =$$

$$= \nu^{2+\theta}\left(\sum_{k=-\infty}^{+\infty} k^2\hat{a}_k^2 - \sum_{|k|\leqslant\nu^{-1+\theta}} k^2\hat{a}_k^2 - \sum_{|k|>\nu^{-1-\frac{\theta}{2}}} k^2\hat{a}_k^2\right) \geqslant$$

$$\geqslant \tilde{c}_1\nu^{1+\theta}\left(1 - \tfrac{2V+\widetilde{C}_2}{\tilde{c}_1}\nu^\theta\right) \geqslant \frac{\tilde{c}_1}{2}\nu^{1+\theta}\,.$$

Thus we have $\hat{A}_{1,\theta}^2 \geqslant \frac{\frac{1}{2}\tilde{c}_1\nu^{1+\theta}}{2\nu^{-1-\theta}} = \frac{\tilde{c}_1}{4}\nu^{2+2\theta}$ and estimate (2.1.8) is proved.

Chapter 3

The multidimensional case

3.1 Introduction

We consider the dynamics of an m-dimensional vector field $\boldsymbol{u} = \boldsymbol{u}(t,\boldsymbol{x})$ on the n-dimensional torus $\mathbb{T}^n = {}^{\mathbb{R}^n}\!/_{\ell\mathbb{Z}^n}$ described by the equation

$$\partial_t\boldsymbol{u} + \nabla_{\boldsymbol{f}(\boldsymbol{u})}\boldsymbol{u} = \nu\Delta\boldsymbol{u} + \boldsymbol{h}(t,\boldsymbol{x})\,. \tag{3.1.1}$$

Here ν is a positive parameter, $\boldsymbol{f}:\mathbb{R}^m \to \mathbb{R}^n$ is a smooth map, $\nabla_{\boldsymbol{f}}$ is the derivative on the direction $\boldsymbol{f}$, and $\boldsymbol{h}$ is a forcing term.

If $m = n$ and $\boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{u}$ we have the usual forced Burgers equation. In a potential case (i.e., the initial state $\boldsymbol{u}_0(\boldsymbol{x}) = \boldsymbol{u}(0,\boldsymbol{x})$ and the field $\boldsymbol{h}$ are gradients of some functions) this can be reduced to a linear parabolic equation.

As it is shown in [1], [18] appropriate bounds for derivatives imply estimates for averaged spectral characteristics of the flow. The purpose of this Chapter is to obtain such bounds for solutions of the Cauchy problem for the generalised m-n multidimensional Burgers equation (3.1.1).

We denote

$$H(t) = \max_{i=1\ldots m}\int_0^t \sup_{\boldsymbol{x}\in\mathbb{T}^n} |h_i(\tau,\boldsymbol{x})|d\tau \tag{3.1.2}$$

and assume that $H(t)$ is finite for finite t. By $|\boldsymbol{u}|$ we denote the following function of t:

$$|\boldsymbol{u}| = |\boldsymbol{u}(t,\cdot)| = \max_{i=1\ldots m} |u_i(t,\cdot)|_{L_\infty}\,,$$

and $|\boldsymbol{u}_0| = |\boldsymbol{u}(0,\cdot)|$. Similarly, $|\boldsymbol{h}| = |\boldsymbol{h}(t,\cdot)| = \max_{i=1\ldots m} |h_i(t,\cdot)|_{L_\infty}$.

We also use the following notation

$$[\boldsymbol{f}]_{C^k(r)} = \max_{j=1\ldots n,\, |\beta|=k} \sup_{\{\boldsymbol{u}\in\mathbb{R}^m : \max|u_i|\leqslant r\}} \left|\frac{\partial^k}{\partial \boldsymbol{u}^\beta} f_j\right| .$$

We define

$$\|\boldsymbol{u}(t,\cdot)\|_k^2 = \int_{\mathbb{T}^n} \sum_{i=1}^m u_i(-\Delta)^k u_i d\boldsymbol{x} = \sum_{i=1}^m \sum_{|\alpha|=k} \tbinom{|\alpha|}{\alpha} \left|D^\alpha u_i\right|^2_{L_2(\mathbb{T}^n)}.$$

Here $\binom{|\alpha|}{\alpha} = \binom{|\alpha|}{\alpha_1,\ldots,\alpha_n} = \frac{(\alpha_1+\cdots+\alpha_n)!}{\alpha_1!\cdots\alpha_n!}$. These are the coefficients in the generalised binomial equation:

$$(x_1 + x_2 + \ldots + x_n)^k = \sum_{|\alpha|=k} \tbinom{k}{\alpha} \boldsymbol{x}^\alpha .$$

We note that $\sum_{|\alpha|=k} \binom{k}{\alpha} = n^k$.

The main results are:

Theorem 3.1.1. *For any $k \geqslant 0$, $t \geqslant 0$, and $\nu > 0$ we have*

$$\|\boldsymbol{u}(t,\cdot)\|_k \leqslant R_k(t) \max\left\{\tfrac{1}{\nu^k}, \tfrac{\|\boldsymbol{u}_0\|_k}{R_k(0)}, \tfrac{\sup_{[0,t]}\|\boldsymbol{h}\|_{k-1}}{R_k(0)}\right\}, \tag{3.1.3}$$

here $\|\boldsymbol{h}\|_{-1} = 0$ and

$$R_k(t) = 4^{k^3} n^{k^2+k} (m+k)^{k^2} m^{\frac{1}{2}} \big(1+\max_{s=0\ldots k}\{[\boldsymbol{f}]_{C^s(|\boldsymbol{u}_0|+H(t))}\,(|\boldsymbol{u}_0|+H(t))^s\}\big)^k (|\boldsymbol{u}_0|+H(t))\ell^{n/2}.$$

Theorem 3.1.2. *Suppose that the initial state $\boldsymbol{u}_0$ is a non-degenerate vector field; then there exist ν-independent positive real constants T and c, and ν-independent positive real constants r_2, r_3, r_4, ... such that:*

If $H(T) < \frac{c}{2}$ then $\forall \nu > 0$ and $\forall k \geqslant 2$, we have:

$$\max_{\substack{j=1\ldots n\\ i=1\ldots m}} \frac{1}{T}\int_0^T \sup_{\boldsymbol{x}\in\mathbb{T}^n} \left|\tfrac{\partial^k u_i}{\partial {x_j}^k}(t,\boldsymbol{x})\right| dt \geqslant \frac{r_k}{\nu^{k/2}} . \tag{3.1.4}$$

Here we have used the following definition:

Definition 3.1.1. *The vector field $\boldsymbol{u}_0$ is degenerate w.r.t. equation (3.1.1) if the matrix $\frac{\partial \boldsymbol{f}(\boldsymbol{u}_0)}{\partial \boldsymbol{x}}$ (this is an $n \times n$ matrix, which depends on $\boldsymbol{x}$) is everywhere nilpotent, i.e., for each point $\boldsymbol{x}$ some power of this matrix is equal to 0.*

The assumption of non-degeneracy is important for Theorem 3.1.2. In Section 3.3 we give an example of a degenerate non-constant initial state for which the derivatives of the solution are bounded by ν-independent constants for all $t \geqslant 0$. Moreover, for the two dimensional case ($m = n = 2$, $\boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{u}$, $\boldsymbol{h} \equiv 0$), we show that any solution with a degenerate initial state retains bounded derivatives. This fact is based on a result from the classical geometry due to Pogorelov – Hartman – Nirenberg, known as the "Cylinder Theorem". In Section 3.3 we also show that in the case $m = n$ and $\boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{u}$ any non-constant *potential* initial state is non-degenerate.

The exponents of viscosity ν in inequalities (3.1.3) and (3.1.4) are not sharp. In the one-dimensional case ($m = n = 1$) we can obtain sharp values for the exponents. In [1] it is shown that for $k \geqslant 1$, we have

$$\|\boldsymbol{u}\|_k \leqslant C_k \left(\tfrac{1}{\nu}\right)^{k-1/2} , \quad \left(\frac{1}{T}\int_0^T \|\boldsymbol{u}\|_k^2 dt\right)^{1/2} \geqslant c_k \left(\tfrac{1}{\nu}\right)^{k-1/2}.$$

As a consequence of these inequalities one can easily get bounds for magnitudes of the derivatives:

$$[\boldsymbol{u}]_k \leqslant \frac{C'_k}{\nu^k} , \quad \frac{1}{T}\int_0^T [\boldsymbol{u}]_k \, dt \geqslant \frac{c'_k}{\nu^k} .$$

This Chapter is organised as follows. In Section 3.2 we prove the upper estimates (3.1.3) (Theorem 3.1.1). We note that they also imply existence of a solution of the Cauchy problem for equation (3.1.1). Section 3.3 is devoted to proving the lower bounds (3.1.4) (Theorem 3.1.2). Here we also discuss different equivalent forms of degeneracy. In Section 3.4 we outline the information concerning the behaviour of the Fourier modes that can be extracted from the bounds (3.1.3) and (3.1.4). We also provide a general tool for this. We discuss how this relates to the Kolmogorov – Obukhov spectral law. In section 3.5 we obtain lower bounds for derivatives of solutions of the Navier Stokes System.

3.2 Upper estimates

In this section we prove Theorem 3.1.1. The componentwise representation of (3.1.1) is

$$\frac{\partial}{\partial t}u_i + \sum_{j=1}^{n} f_j(u_1,\dots,u_m)\frac{\partial u_i}{\partial x_j} = \nu\Delta u_i + h_i(t,x_1,\dots,x_n)\,, \tag{3.2.1}$$

where $i = 1,\dots,m$. From the maximum principle, we have

$$|\boldsymbol{u}(t,\cdot)| \leqslant |\boldsymbol{u}_0| + H(t)\,, \tag{3.2.2}$$

where $H(t)$ is defined by (3.1.2). We actually obtain (3.1.3) for $k = 0$:

$$\|\boldsymbol{u}(t,\cdot)\|_0 \leqslant \sqrt{m}\ell^{n/2}(|\boldsymbol{u}_0| + H(t))\,. \tag{3.2.3}$$

We multiply (3.2.1) by $(-\Delta)^k u_i$, take the sum over $i = 1,\dots,m$, and integrate over the period (over the torus):

$$\tfrac{1}{2}\tfrac{d}{dt}\|\boldsymbol{u}\|_k^2 - \Upsilon_1 = -\nu\|\boldsymbol{u}\|_{k+1}^2 + \Upsilon_2\,,$$

where we denote

$$\Upsilon_1 = -\int_{\mathbb{T}^n} \sum_{\substack{j=1\dots n\\ i=1\dots m}} f_j(u_1,\dots,u_m)\frac{\partial u_i}{\partial x_j} \sum_{\kappa_1,\dots,\kappa_k=1\dots n} (-1)^k \frac{\partial^2}{\partial x_{\kappa_1}^2}\cdots\frac{\partial^2}{\partial x_{\kappa_k}^2}u_i d\boldsymbol{x} \tag{3.2.4}$$

and

$$\Upsilon_2 = \int_{\mathbb{T}^n} \sum_{i=1\dots m} h_i(-\Delta)^k u_i d\boldsymbol{x}\,.$$

Lemma.

$$\Upsilon_1 \leqslant 4^{k^2} n^{k+1}\tfrac{(m+k-1)!}{(m-1)!} \max_{s=0\dots k}\{[\boldsymbol{f}]_{C^s(|\boldsymbol{u}|)}\,|\boldsymbol{u}|_{L_\infty}^s\}\,\|\boldsymbol{u}\|_k\,\|\boldsymbol{u}\|_{k+1}\,. \tag{3.2.5}$$

Proof. The right hand side of (3.2.4) contains nmn^k terms. Take any one of them and integrate it by parts $k-1$ times:

$$term = \int_{\mathbb{T}^n} \frac{\partial}{\partial x_{\kappa_2}}\cdots\frac{\partial}{\partial x_{\kappa_k}}\left(f_j(u_1,\dots,u_m)\frac{\partial u_i}{\partial x_j}\right)\frac{\partial^2}{\partial x_{\kappa_1}^2}\frac{\partial}{\partial x_{\kappa_2}}\cdots\frac{\partial}{\partial x_{\kappa_k}}u_i d\boldsymbol{x}\,.$$

Expanding the brackets we get no more than $(m+1)(m+2)\cdots(m+k-1)$ terms of the form

$$\int_{\mathbb{T}^n} D_{\boldsymbol{u}}^{\beta} f_j D_{\boldsymbol{x}}^{\alpha_1} u_{\#}\cdots D_{\boldsymbol{x}}^{\alpha_{|\beta|+1}} u_{\#}\frac{\partial^2}{\partial x_{\kappa_1}^2}\frac{\partial}{\partial x_{\kappa_2}}\cdots\frac{\partial}{\partial x_{\kappa_k}}u_i d\boldsymbol{x}\,,$$

where $|\alpha_1| + \dots + |\alpha_{|\beta|+1}| = k$ and #'s stand for various indexes between 1 and m. The modulus of this integral is not bigger than

$$\mathfrak{A} = \left|D_{\boldsymbol{u}}^{\beta} f_j\right|_{L_\infty(B(|\boldsymbol{u}|))}\left|D_{\boldsymbol{x}}^{\alpha_1} u_{\#}\right|_{L_{\frac{2k}{|\alpha_1|}}}\cdots\left|D_{\boldsymbol{x}}^{\alpha_{|\beta|+1}} u_{\#}\right|_{L_{\frac{2k}{|\alpha_{|\beta|+1}|}}}\|\boldsymbol{u}\|_{k+1}\,.$$

Here $B(r)$ denotes the ball in $\mathbb{R}^m$ of radius r in the l_∞ norm, centred at the origin. Using the Gagliardo-Nirenberg inequality (see [13], pp. 106-107)

$$|D_{\boldsymbol{x}}^{\alpha_i} u_s|_{L_{\frac{2k}{|\alpha_i|}}} \leqslant 4^{|\alpha_i|(k-|\alpha_i|)}|u_s|_{L_\infty}^{1-\frac{|\alpha_i|}{k}}\|u_s\|_k^{\frac{|\alpha_i|}{k}}\,,$$

we obtain

$$\mathfrak{A} \leqslant 4^{k^2}|D_{\boldsymbol{u}}^{\beta} f_j|_{L_\infty(B(|\boldsymbol{u}|))}|\boldsymbol{u}|_{L_\infty}^{|\beta|}\,\|\boldsymbol{u}\|_k\,\|\boldsymbol{u}\|_{k+1}\,.$$

Eventually we have $\Upsilon_1 \leqslant n^{k+1}m(m+1)\cdots(m+k-1)\max\{\mathfrak{A}\}$. □

Corollary. For $k \geqslant 1$ we have:

$$\Upsilon_1 \leqslant B_k(t)\,\|\boldsymbol{u}\|_k\,\|\boldsymbol{u}\|_{k+1}\,, \tag{3.2.6}$$

where $B_k(t) = 4^{k^2}n^{k+1}(m+k)^k\big(1 + \max\limits_{s=0\dots k}\{[\boldsymbol{f}]_{C^s(|\boldsymbol{u}_0|+H(t))}\,(|\boldsymbol{u}_0| + H(t))^s\}\big) - 1$.

Indeed, using inequality (3.2.2), we see that for $k \geqslant 1$ the r.h.s. of (3.2.6) is bigger than the r.h.s. of (3.2.5).

Integrating Υ_2 by parts, we obtain $\Upsilon_2 \leqslant C_k\,\|\boldsymbol{u}\|_{k+1}$, where $C_k = \sup\limits_t \|\boldsymbol{h}\|_{k-1}$. So we have

$$\tfrac{1}{2}\tfrac{d}{dt}\,\|\boldsymbol{u}\|_k^2 \leqslant \|\boldsymbol{u}\|_{k+1}\,(-\nu\,\|\boldsymbol{u}\|_{k+1} + B_k(t)\,\|\boldsymbol{u}\|_k + C_k)\,. \tag{3.2.7}$$

Now using the inequality $\|\boldsymbol{u}\|_{k+1} \geqslant \|\boldsymbol{u}\|_k\left(\frac{\|\boldsymbol{u}\|_k}{\|\boldsymbol{u}\|_0}\right)^{1/k}$, we have

$$\tfrac{1}{2}\tfrac{d}{dt}\,\|\boldsymbol{u}\|_k^2 \leqslant \|\boldsymbol{u}\|_{k+1}\left(\|\boldsymbol{u}\|_k\left(-\nu\big(\tfrac{\|\boldsymbol{u}\|_k}{\|\boldsymbol{u}\|_0}\big)^{1/k} + B_k(t)\right) + C_k\right).$$

It follows from this that

$$\text{if } \|\boldsymbol{u}\|_k > \frac{(B_k(t)+1)^k\,\|\boldsymbol{u}\|_0}{\nu^k} \text{ and } \|\boldsymbol{u}\|_k > C_k, \text{ then } \|\boldsymbol{u}\|_k \text{ is decreasing.} \tag{3.2.8}$$

We denote the r.h.s. of (3.1.3) by $F_k(t)$. It is clear from the definition of the function F_k that

$$\|\boldsymbol{u}(0,\cdot)\|_k \leqslant F_k(0).$$

Using (3.2.3) we see that if $\|\boldsymbol{u}\|_k > F_k(t)$ then $\|\boldsymbol{u}\|_k$ is decreasing by argument (3.2.8). Since $F_k(t)$ is a non-decreasing function we obtain that $\|\boldsymbol{u}\|_k$ can never be greater than $F_k(t)$. We arrive at (3.1.3). Theorem 3.1.1 is proven.

3.3 Lower estimates

In this section we prove Theorem 3.1.2. Throughout this section we use standard facts from linear algebra about linear transformations. For the convenience of the reader we very briefly outline the proofs. See reference [9], which covers these in an elegant, coordinate free way. We start from brief discussing of the notion of the degeneracy of a vector field.

3.3.1 Degeneracy condition

Using the fact that an $n \times n$ matrix A is nilpotent iff $A^n = 0$, we can give a definition of degeneracy that is equivalent to the previous one, but more robust.

Definition 3.3.1. *The vector field $\boldsymbol{u}_0$ is degenerate iff* $\left(\frac{\partial \boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{x}))}{\partial \boldsymbol{x}}\right)^n \equiv 0$.

Let $\tilde{\boldsymbol{f}}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{x}))$. Consider the characteristic polynomial of the matrix $\frac{\partial \tilde{\boldsymbol{f}}}{\partial \boldsymbol{x}}$:

$$\chi(\lambda) = \det\big(\tfrac{\partial \tilde{\boldsymbol{f}}}{\partial \boldsymbol{x}} - \lambda \mathbf{1}\big) = (-\lambda)^n + (-\lambda)^{n-1} I_1(\boldsymbol{x}) + \cdots + I_n(\boldsymbol{x})\,.$$

Expanding the determinant, we obtain

$$I_k(\boldsymbol{x}) = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} \det \begin{pmatrix} \frac{\partial \tilde{f}_{i_1}}{\partial x_{i_1}} & \cdots & \frac{\partial \tilde{f}_{i_1}}{\partial x_{i_k}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \tilde{f}_{i_k}}{\partial x_{i_1}} & \cdots & \frac{\partial \tilde{f}_{i_k}}{\partial x_{i_k}} \end{pmatrix}. \tag{3.3.1}$$

Using the Jordan form of the matrix we obtain that if $\frac{\partial \tilde{f}_i}{\partial x_j}(\boldsymbol{x})$ is nilpotent then all $I_j(\boldsymbol{x})$ are zeros. From the Hamilton-Cayley identity (any matrix is a root of its characteristic polynomial) we get the converse. Thus we have that the matrix $\frac{\partial \tilde{f}_i}{\partial x_j}(\boldsymbol{x})$ is nilpotent iff $I_1(\boldsymbol{x}) = \cdots = I_n(\boldsymbol{x}) = 0$. Due to this we give another equivalent definition of degeneracy which we will use subsequently.

Definition 3.3.2. *The vector field $\boldsymbol{u}_0$ is degenerate iff each $I_k(\boldsymbol{x})$ above, are identically equal to zero.*

Also note that if $m < n$, then the matrix $\frac{\partial \tilde{f}_i}{\partial x_j}$ has rank $\leqslant m$, so for $k \in [m+1, n]$ we have $I_k(\boldsymbol{x}) \equiv 0$.

Lemma 3.3.1. *For each $k = 1, \ldots, n$ we have $\int_{\mathbb{T}^n} I_k(\boldsymbol{x}) d\boldsymbol{x} = 0$.*

We give two proofs of this fact. The first proof is geometrical. This proof is concise, but is hard to follow without certain concepts from the differential geometry. The second proof is more cumbersome, but does not require any special knowledge.

Proof 1. We need to show that

$$\int_{\mathbb{T}^n} \det\big(\tfrac{\partial \tilde{f}_i}{\partial x_j}(\boldsymbol{x}) + \lambda \mathbf{1}\big) d\boldsymbol{x} \equiv (\ell\lambda)^n. \tag{3.3.2}$$

Since both the l.h.s and the r.h.s. are polynomials in λ, it is sufficient to prove this equality for all integer λ. We write $\frac{\partial \tilde{f}_i}{\partial x_j}(\boldsymbol{x}) + \lambda \mathbf{1} = \frac{\partial \boldsymbol{\Psi}}{\partial \boldsymbol{x}}$, where the vector valued function $\boldsymbol{\Psi}$ is defined by the formula $\boldsymbol{\Psi}(\boldsymbol{x}) = \tilde{\boldsymbol{f}}(\boldsymbol{x}) + \lambda \boldsymbol{x}$. Since λ is an integer, then this function defines a map from $\mathbb{T}^n$ to $\mathbb{T}^n$. Since $\boldsymbol{\Psi}$ is homotopic to the map $\boldsymbol{x} \mapsto \lambda \boldsymbol{x}$ on torus $\mathbb{T}^n$ (a homotopy is given by $\boldsymbol{\Psi}_t(\boldsymbol{x}) = t\tilde{\boldsymbol{f}}(\boldsymbol{x}) + \lambda \boldsymbol{x}$), we have $\deg \boldsymbol{\Psi} = \deg\{\boldsymbol{x} \mapsto \lambda \boldsymbol{x}\}$ and hence $\deg \boldsymbol{\Psi} = \lambda^n$. Using the formula

$$\int_{\mathbb{T}^n} \det \tfrac{\partial \boldsymbol{\Psi}}{\partial \boldsymbol{x}} d\boldsymbol{x} = \deg \boldsymbol{\Psi} \int_{\mathbb{T}^n} d\boldsymbol{x}$$

(see [7], II, chpt. 3) we arrive at (3.3.2) (since $\int_{\mathbb{T}^n} d\boldsymbol{x} = \ell^n$). □

Proof 2. We show that the integral over the torus $\mathbb{T}^n$ of each determinant in (3.3.1) vanishes. We show that the integral of the first term in (3.3.1) is zero, i.e., $\int_{\mathbb{T}^n} \frac{\partial(f_1, \ldots, f_k)}{\partial(x_1, \ldots, x_k)} d\boldsymbol{x} = 0$. This holds for all the other terms in (3.3.1), and this can be proved in exactly the same way, since these terms are identical up to the selection of indexes.

For $k=1$ this is obvious, so we consider the case $k \geqslant 2$. Integrating by parts we have:

$$\int_{\mathbb{T}^n} \tfrac{\partial(f_1,\dots,f_k)}{\partial(x_1,\dots,x_k)} d\boldsymbol{x} = \int_{\mathbb{T}^n} \sum_{\sigma\in S_k} \operatorname{sign}\sigma \, \tfrac{\partial f_1}{\partial x_{\sigma(1)}} \cdots \tfrac{\partial f_k}{\partial x_{\sigma(k)}} d\boldsymbol{x} =$$

$$- \int_{\mathbb{T}^n} \sum_{j=2}^{k} \sum_{\sigma\in S_k} \operatorname{sign}\sigma \, f_1 \tfrac{\partial f_2}{\partial x_{\sigma(2)}} \cdots \tfrac{\partial^2 f_j}{\partial x_{\sigma(1)} \partial x_{\sigma(j)}} \cdots \tfrac{\partial f_k}{\partial x_{\sigma(k)}} d\boldsymbol{x} =$$

$$- \int_{\mathbb{T}^n} \sum_{j=2}^{k} \sum_{\sigma\in S_k} J_{j,\sigma}(\boldsymbol{x}) d\boldsymbol{x}.$$

This sum contains $k!\,(k-1)$ terms parametrised by $\sigma \in S_k$ and $j \in \{2,\dots,k\}$. The remainder of the proof follows from the fact that

$$J_{j,\sigma} = -J_{j,\sigma_{(1\leftrightarrow j)}},$$

where the permutation $\sigma_{(1\leftrightarrow j)}$ is the product of the transposition $(1 \leftrightarrow j)$ and the permutation σ:

$$\sigma_{(1\leftrightarrow j)}(1) = \sigma(j)\,, \quad \sigma_{(1\leftrightarrow j)}(j) = \sigma(1)\,, \quad \sigma_{(1\leftrightarrow j)}(i) = \sigma(i) \ \text{ for } i \neq 1, j. \qquad \square$$

It follows that any potential degenerate initial state is constant. Indeed, if $\boldsymbol{f}(\boldsymbol{u}_0) = \nabla U$ then the function $U : \mathbb{R}^n \to \mathbb{R}$ necessarily has no more than linear growth (because $\boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{x}))$ is periodic) and is also a harmonic function (because $\Delta U(\boldsymbol{x}) = \operatorname{div} \boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{x})) = I_1(\boldsymbol{x}) \equiv 0$); so $U(\boldsymbol{x}) = B\boldsymbol{x} + \boldsymbol{c}$ where B and $\boldsymbol{c}$ are a constant matrix and a constant vector respectively.

Consider the case $n = 2$, i.e., $\dim \boldsymbol{x} = 2$.

Theorem 3.3.2. *If $n = 2$ then any degenerate initial state has the form:*

$$\begin{aligned} \big\{\boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{x}))\big\}_1 &= \quad b_2\varphi_0(b_1x_1 + b_2x_2) + c_1 \\ \big\{\boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{x}))\big\}_2 &= -b_1\varphi_0(b_1x_1 + b_2x_2) + c_2 \end{aligned} \tag{3.3.3}$$

for a suitable function $\varphi_0(\cdot)$ and real numbers b_1, b_2, c_1, and c_2.

Proof. Let c_1 and c_2 be the mean values of $f_1(\boldsymbol{u}_0(\boldsymbol{x}))$ and $f_2(\boldsymbol{u}_0(\boldsymbol{x}))$, respectively. Since $\operatorname{div} \boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{x})) = 0$, there exists a function $\psi : \mathbb{R}^2 \to \mathbb{R}$ such that $\operatorname{rot}\psi = \boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{x})) - \binom{c_1}{c_2}$, where $\operatorname{rot}\psi = \binom{\partial\psi/\partial x_2}{-\partial\psi/\partial x_1}$. We note that the function $\psi(x_1, x_2)$ is $\mathbb{T}^2$-periodic, and hence bounded. Since the determinant of the Jacobi matrix of $\boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{x}))$ is zero, we have that determinant of the Hessian of ψ is zero. Consider the graph of the function ψ in $\mathbb{R}^3$. The Gaussian curvature of this surface is given by the formula (see [7], I, chpt. 2)

$$K = \frac{\psi_{xx}\psi_{yy} - \psi_{xy}^2}{(1 + \psi_x^2 + \psi_y^2)^2}.$$

Therefore, $K = 0$. Now we use the fact that any complete surface of constant zero Gaussian curvature is a cylinder over a flat curve (see [30]; [32], chpt. 5). Since the function ψ is bounded we have that any element of this cylinder is a horizontal line, and hence has the form

$$\begin{cases} b_1x_1 + b_2x_2 = const, \\ z = \tilde{\psi}(const). \end{cases}$$

We conclude that

$$\psi(x_1, x_2) = \tilde{\psi}(b_1x_1 + b_2x_2)$$

and (3.3.3) follows with $\varphi_0 = \tilde{\psi}'$. $\square$

Corollary. *Suppose, $m = n = 2$, $\boldsymbol{f}(\boldsymbol{u}) \equiv \boldsymbol{u}$, and $\boldsymbol{h} \equiv 0$; then the solution of the Cauchy problem (3.1.1), (3.3.3) remains of the form (3.3.3):*

$$\boldsymbol{u}(t,\boldsymbol{x}) = \begin{pmatrix} b_2 \\ -b_1 \end{pmatrix} \varphi(t, b_1x_1 + b_2x_2) + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{3.3.4}$$

where the function φ satisfies to the following equation

$$\varphi_t + (b_1c_1 - b_2c_2)\varphi' = (b_1^2 + b_2^2)\nu\varphi''\,. \tag{3.3.5}$$

In this case we have ν-independent upper bounds for the derivatives of the solution.

Further on we shall use the polynomial

$$P(t) = t^n\chi(\tfrac{-1}{t}) = \det\big(\delta_{ij} + \tfrac{\partial \tilde{f}_i}{\partial x_j} t\big) = 1 + I_1(\boldsymbol{x})t + I_2(\boldsymbol{x})t^2 + \cdots + I_n(\boldsymbol{x})t^n, \tag{3.3.6}$$

rather than a characteristic polynomial.

3.3.2 General idea

In this subsection we present an auxiliary theorem from which we then derive Theorem 3.1.2. This auxiliary theorem is technically complicated. Here we deal mainly with general ideas, and postpone the technicalities to the next subsection.

We denote the right hand side of (3.2.1) by g_i:

$$\frac{\partial}{\partial t}u_i + \sum_{j=1}^{n} f_j(u_1,\dots,u_m)\frac{\partial u_i}{\partial x_j} = g_i(t,x_1,\dots,x_n)\,. \tag{3.3.7}$$

Theorem 3.3.3. *Let $\boldsymbol{u}_0$ be non-degenerate. Then there exist $T = T(\boldsymbol{f},\boldsymbol{u}_0)$ and $c = c(\boldsymbol{f},\boldsymbol{u}_0)$ such that*

$$\max_{i=1\dots m}\int_0^T \sup_{\boldsymbol{x}\in\mathbb{T}^n} |g_i(\tau,\boldsymbol{x})|d\tau \geqslant c\,. \tag{3.3.8}$$

Proof. Consider the flow on the cylinder $\mathbb{T}^n\times[0,\infty)$ generated by the vector field $\boldsymbol{f}(\boldsymbol{u})$. In other words we consider the Cauchy problem

$$\tfrac{d}{dt}\boldsymbol{\gamma}(t,\boldsymbol{\xi}) = \boldsymbol{f}(\boldsymbol{u}(t,\boldsymbol{\xi}))$$

with the initial state $\boldsymbol{\gamma}(0,\boldsymbol{\xi}) = \boldsymbol{\xi}$. Here $\boldsymbol{\xi}$ is the Lagrange coordinate of the flow $\boldsymbol{\gamma}$.

For any fixed time T we have a map $\boldsymbol{\gamma}(T,\cdot):\mathbb{T}^n\to\mathbb{T}^n$. It is known that Jacobian of such a map is positive at any point. (See [10].)

Combining the chain rule and (3.3.7), we obtain

$$\tfrac{d}{dt}\boldsymbol{u}(t,\boldsymbol{\gamma}(t,\boldsymbol{\xi})) = \boldsymbol{g}(t,\boldsymbol{\gamma}(t,\boldsymbol{\xi}))\,.$$

Suppose $\boldsymbol{g}(\cdot)\equiv 0$; then $\boldsymbol{f}\big(\boldsymbol{u}(t,\boldsymbol{\gamma}(t,\boldsymbol{\xi}))\big)\equiv\boldsymbol{f}\big(\boldsymbol{u}(0,\boldsymbol{\gamma}(0,\boldsymbol{\xi}))\big)$ and

$$\boldsymbol{\gamma}(t,\boldsymbol{\xi}) = \boldsymbol{\gamma}^{\boldsymbol{0}}(t,\boldsymbol{\xi}) = \boldsymbol{\xi} + t\boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{\xi}))\,. \tag{3.3.9}$$

It follows that if the function $\boldsymbol{g}$ is small, then the flow $\boldsymbol{\gamma}(t,\boldsymbol{\xi})$ is close (in the C^0-norm) to the right hand side of (3.3.9). For a detailed proof of this fact, refer to the next subsection. For the time being we simply note that this is a consequence of the following inequality:

$$\big|u_i(t,\boldsymbol{\gamma}(t,\boldsymbol{\xi})) - u_i(0,\boldsymbol{\gamma}(0,\boldsymbol{\xi}))\big| \leqslant \int_0^t \sup_{\boldsymbol{x}\in\mathbb{T}^n}|g_i(\tau,\boldsymbol{x})|d\tau\,.$$

Since for each $k=1,\dots,n$, $\int_{\mathbb{T}^n} I_k(\boldsymbol{x})d\boldsymbol{x} = 0$ (see lemma 3.3.1) and some of the I_k are not identically zero (due to the non-degeneracy of $\boldsymbol{u}_0$), we obtain that there exists

a point $\boldsymbol{x}^\star\in\mathbb{T}^n$ and a number $l\in[1,\dots,n]$ such that $I_l(\boldsymbol{x}^\star)<0$ and $I_k(\boldsymbol{x}^\star)=0$ for $k>l$.

The Jacobian of (3.3.9) is expressed by polynomial (3.3.6). For the time $t=T$ at the point $\boldsymbol{\xi}=\boldsymbol{x}^\star$, we have

$$\frac{\partial\boldsymbol{\gamma}^0}{\partial\boldsymbol{\xi}} = P(T) = 1 + I_1(\boldsymbol{x}^\star)T + I_2(\boldsymbol{x}^\star)T^2 + \dots + I_l(\boldsymbol{x}^\star)T^l\,.$$

We take large enough time T such that this Jacobian is negative. Suppose that

$$\max_{i=1\dots m}\int_0^T \sup_{\boldsymbol{x}\in\mathbb{T}^n}|g_i(\tau,\boldsymbol{x})|d\tau < c$$

and c is a small enough number; then the map $\boldsymbol{\gamma}(T,\cdot)$ is close to the map $\boldsymbol{\gamma}^0(T,\cdot)$ with negative Jacobian at the point $\boldsymbol{x}^\star$. Taking c small enough we have a contradiction with the positivity of the Jacobian of the map $\boldsymbol{\gamma}(T,\cdot)$. (See the next part of this section for more details). □

Let us turn to equation (3.1.1). Suppose $H(t)<\frac{c}{2}$; then we obtain

$$\max_{i=1\dots m}\int_0^T \sup_{\boldsymbol{x}\in\mathbb{T}^n}|\nu\Delta u_i(\tau,\boldsymbol{x})|\,d\tau > \frac{c}{2}\,.$$

Hence we have inequality (3.1.4) for $k=2$ with the constant $r_2=\frac{c}{2nT}$. We now fix the indexes i and j to the values for which the maximum in (3.1.4) for $k=2$ is achieved. For these indexes we have

$$\frac{1}{T}\int_0^T \sup_{\boldsymbol{x}\in\mathbb{T}^n}\big|\tfrac{\partial^2 u_i}{\partial x_j^2}(\tau,\boldsymbol{x})\big|d\tau \geqslant \tfrac{c}{2nT}\,\tfrac{1}{\nu}\,. \tag{3.3.10}$$

Using the interpolation inequality

$$\big|\tfrac{\partial^2 u_i}{\partial x_j^2}(\tau,\cdot)\big|_{L_\infty} \leqslant C_{k,2}\,|u_i(\tau,\cdot)|_{L_\infty}^{\frac{k-2}{k}}\,\big|\tfrac{\partial^k u_i}{\partial x_j^k}(\tau,\cdot)\big|_{L_\infty}^{\frac{2}{k}} \tag{3.3.11}$$

(see [14], where the best constants are also obtained) and inequality (3.2.2), we obtain

$$\big|\tfrac{\partial^k u_i}{\partial x_j^k}(\tau,\cdot)\big|_{L_\infty} \geqslant \frac{\big|\tfrac{\partial^2 u_i}{\partial x_j^2}(\tau,\cdot)\big|_{L_\infty}^{k/2}}{\sqrt{3}\big(|\boldsymbol{u}_0| + H(\tau)\big)^{\frac{k-2}{2}}}\,. \tag{3.3.12}$$

Here we have used the inequality $(C_{k,2})^{k/2}\leqslant\sqrt{3}$, which is proved in the Appendix.

Integrating (3.3.12) we obtain

$$\frac{1}{T}\int_0^T \left|\tfrac{\partial^k u_i}{\partial x_j^k}(\tau,\cdot)\right|_{L_\infty} d\tau \geqslant \frac{\frac{1}{T}\int_0^T \left|\frac{\partial^2 u_i}{\partial x_j^2}(\tau,\cdot)\right|_{L_\infty}^{k/2} d\tau}{\sqrt{3}\big(|\boldsymbol{u}_0|+H(T)\big)^{\frac{k-2}{2}}} \geqslant$$
$$\frac{\big(\frac{1}{T}\int_0^T \left|\frac{\partial^2 u_i}{\partial x_j^2}(\tau,\cdot)\right|_{L_\infty} d\tau\big)^{k/2}}{\sqrt{3}\big(|\boldsymbol{u}_0|+H(T)\big)^{\frac{k-2}{2}}} \overset{(3.3.10)}{\geqslant} \frac{\big(\frac{c}{2nT\nu}\big)^{k/2}}{\sqrt{3}\big(|\boldsymbol{u}_0|+H(T)\big)^{\frac{k-2}{2}}}\,.$$

This concludes the proof of (3.1.4) for $k>2$ with the constants

$$r_k = \frac{(\frac{c}{2nT})^{\frac{k}{2}}}{\sqrt{3}\big(|\boldsymbol{u}_0|+H(T)\big)^{\frac{k-2}{2}}}\,.$$

In the next subsection we will specify the values of T and c (see (3.3.17) and (3.3.37) respectively).

3.3.3 Technicalities

In this subsection we introduce a more general approach to Theorem 3.1.2, namely, we consider the non-periodic case. From the previous part of this subsection we already know that the crucial condition for Theorem 3.1.2 is "negativity" rather than non-degeneracy of the matrix $\frac{\partial \boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{x}))}{\partial \boldsymbol{x}}$. Let $Q=[0,\infty)\times\mathbb{R}^n$. Let $\boldsymbol{u}: Q\to\mathbb{R}^m$ be a C^1-smooth vector-valued function and $\boldsymbol{f}:\mathbb{R}^m\to\mathbb{R}^n$ be a C^1-smooth map such that $\tilde{\boldsymbol{f}}(\boldsymbol{x})=\boldsymbol{f}(\boldsymbol{u}(0,\boldsymbol{x}))$ is a C^2-smooth map $\mathbb{R}^n\to\mathbb{R}^n$. In this subsection $\boldsymbol{x}=(x_1,\dots,x_n)$ are coordinates in $\mathbb{R}^n$.

We define $g_i: Q\to\mathbb{R}$, $(i=1,\dots,m)$ to satisfy:

$$\partial_t u_i + \sum_{j=1}^n f_j(u_1,\dots,u_m)\frac{\partial u_i}{\partial x_j} = g_i(t,x_1,\dots,x_n)\,. \tag{3.3.13}$$

We assume that that the following quantities exist and are finite:

$$|\tilde{\boldsymbol{f}}|_1 = \max_{i,j}\sup_{\boldsymbol{x}}\left|\frac{\partial \tilde f_i}{\partial x_j}\right|, \qquad |\tilde{\boldsymbol{f}}|_2 = \max_{i,j,k}\sup_{\boldsymbol{x}}\left|\frac{\partial^2 \tilde f_i}{\partial x_j\partial x_k}\right|. \tag{3.3.14}$$

We denote by $|\cdot|_{l_\infty}$ the l_∞ norm on $\mathbb{R}^m$ and/or $\mathbb{R}^n$. For any $\boldsymbol{u}\in\mathbb{R}^m$ the derivative $\boldsymbol{\nabla_u f}:\mathbb{R}^m\to\mathbb{R}^n$ is a linear map. We consider the norm of this map associated with

the l_∞-norms on $\mathbb{R}^m$ and $\mathbb{R}^n$. For any domain $E\subset\mathbb{R}^m$ we denote by $\|\boldsymbol{\nabla_u f}\|_E$ the supremum over $\boldsymbol{u}\in E$ of these norms:

$$\|\boldsymbol{\nabla_u f}\|_E = \sup_{\boldsymbol{u}\in E}\max_{|\boldsymbol{v}|_{l_\infty}=1}\big|(\boldsymbol{\nabla_u f})\boldsymbol{v}\big|_{l_\infty}, \tag{3.3.15}$$

where $\{(\boldsymbol{\nabla_u f})\boldsymbol{v}\}_j = \sum\limits_{i=1\dots m}\frac{\partial f_j}{\partial u_i}v_i$. For the quantities (3.3.1) we have:

$$|I_k| \leqslant \tbinom{n}{k} k^{k/2}|\tilde{\boldsymbol{f}}|_1^k\,. \tag{3.3.16}$$

Indeed, the r.h.s. of (3.3.1) contains $\binom{n}{k}$ terms and each of them are no greater than $k^{k/2}|\tilde{\boldsymbol{f}}|_1^k$. Here we have used the fact that the volume of a k-dimensional parallelepiped with sides of length less than or equal to $\sqrt{k}|\tilde{\boldsymbol{f}}|_1$ is no greater than $(\sqrt{k}|\tilde{\boldsymbol{f}}|_1)^k$.

Theorem 3.3.4. *Let ε be a positive real number and $l\in\{1,2,\dots,n\}$. Suppose that there exists $\boldsymbol{x}^\star\in\mathbb{R}^n$ such that $I_l(\boldsymbol{x}^\star)=-\varepsilon<0$ and that for $k=l+1,\dots,n$ we have $I_k(\boldsymbol{x}^\star)=0$. Let*

$$T = 2^n l^{l/2}\tfrac{|\tilde{\boldsymbol{f}}|_1^{l-1}}{\varepsilon}\,. \tag{3.3.17}$$

Let c_1 be any positive real number. Suppose

$$\max_{i=1\dots m}\int_0^T \sup_{\boldsymbol{x}\in\mathbb{R}^n}|g_i(\tau,\boldsymbol{x})|\,d\tau < c_1\,, \tag{3.3.18}$$

then

$$\int_0^T (T-\tau)\max_{i=1\dots m}\sup_{\boldsymbol{x}\in\mathbb{R}^n}|g_i(\tau,\boldsymbol{x})|\,d\tau \geqslant c_2\,, \tag{3.3.19}$$

where

$$c_2 = \frac{\frac{1}{2}\,l^l}{n^{\frac{3}{2}(n+1)}(|\tilde{\boldsymbol{f}}|_1 T)^{2n-2l}\,\|\boldsymbol{\nabla_u f}\|_{B(|\boldsymbol{u}_0|+c_1)}|\tilde{\boldsymbol{f}}|_2 T}\,. \tag{3.3.20}$$

Here $B(r)$ denotes the ball in $\mathbb{R}^m$ of radius r in the l_∞ norm, centred at the origin.

Proof. First of all we note that $\boldsymbol{f}(\boldsymbol{u}(t,\boldsymbol{x}))$ and $\frac{\partial}{\partial\boldsymbol{x}}\boldsymbol{f}(\boldsymbol{u}(t,\boldsymbol{x}))$ are a continuous vector function and a continuous matrix function, respectively. It follows from this that

1° $\exists!$ solution of the Cauchy problem for the following ODE in R^n:

$$\tfrac{d}{dt}\boldsymbol{\gamma}(t,\boldsymbol{\xi}) = \boldsymbol{f}(\boldsymbol{u}(t,\boldsymbol{\xi}))\,, \tag{3.3.21}$$

with the initial state $\boldsymbol{\gamma}(0,\boldsymbol{\xi})=\boldsymbol{\xi}\in\mathbb{R}^n$.

2° This solution $\boldsymbol{\gamma} \in C^1\big([0,T]\times\mathbb{R}^n;\mathbb{R}^n\big)$.

3° This solution satisfies $det\left(\frac{\partial\boldsymbol{\gamma}(t,\boldsymbol{\xi})}{\partial\boldsymbol{\xi}}\right) = \exp\int_0^t \sum_{i=0}^n \frac{\partial}{\partial\gamma_i} f_i\big(\boldsymbol{u}(t,\boldsymbol{\gamma}(t,\boldsymbol{\xi}))\big)$, see [10].

The positiveness of the Jacobian in 3° above implies inequality (3.3.19). Indeed, if we assume that (3.3.19) does not hold it follows that this Jacobian is negative somewhere. The rest of this subsection is devoted to proving this.

In the proof of proposition 1 bellow we will use the fact that there exists a continuous second derivative $\frac{d^2}{dt^2}\boldsymbol{\gamma}(t,\boldsymbol{\xi})$. This follows from the existence and continuity of the first partial derivative $\frac{\partial}{\partial t}\boldsymbol{f}(\boldsymbol{u}(t,\boldsymbol{x}))$. In the proof of propositions 2 and 4 below we will use the inequality

$$T|\tilde{\boldsymbol{f}}|_1 > 1\,. \tag{3.3.22}$$

This inequality follows from (3.3.17) and (3.3.16) with $k=l$, since $\varepsilon = |I_l(\boldsymbol{x}^\star)|$.

We fix T at the value given in (3.3.17). For fixed t, $\boldsymbol{\gamma}(t,\boldsymbol{x})$ defines a mapping from $\mathbb{R}^n$ into itself. We consider the case $t=T$ and decompose this mapping as follows:

$$\gamma_i(T,\boldsymbol{x}) = p_i(\boldsymbol{x}) + q_i(\boldsymbol{x})\,. \tag{3.3.23}$$

Here $\boldsymbol{p}(\boldsymbol{x})$ is the zeroth and first terms of the Tailor expansion:

$$p_i(\boldsymbol{x}) = \gamma_i(0,\boldsymbol{x}) + \frac{d}{dt}\gamma_i(0,\boldsymbol{x})T = \boldsymbol{x} + \tilde{\boldsymbol{f}}(\boldsymbol{x})T\,. \tag{3.3.24}$$

The remainder of the Tailor expansion (the vector $\boldsymbol{q}(\boldsymbol{x})\in\mathbb{R}^n$) can be represented as

$$q_i(\boldsymbol{x}) = \int_0^T (T-\tau)\frac{d^2}{dt^2}\gamma_i(\tau,\boldsymbol{x})\,d\tau\,. \tag{3.3.25}$$

Proposition 1. *For the Euclidean norm of the vector $\boldsymbol{q}$ we have:*

$$\|\boldsymbol{q}\| \leqslant \sqrt{n}\,\|\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{f}\|_{B(|\boldsymbol{u}_0|+c_1)}\int_0^T (T-\tau)\max_{i=1\dots m}\sup_{\boldsymbol{x}\in\mathbb{R}^n}|g_i(\tau,\boldsymbol{x})|\,d\tau\,. \tag{3.3.26}$$

Here $\|\boldsymbol{q}\| = \sqrt{q_1^2+\dots+q_n^2}$. We will use the notation $\|\cdot\|$ for the Euclidean norm subsequently.

Proof. Combining the chain rule with (3.3.21) and (3.3.13) we have

$$\frac{d}{dt}u_i(t,\boldsymbol{\gamma}(t,\boldsymbol{x})) = g_i(t,\boldsymbol{\gamma}(t,\boldsymbol{x}))\,.$$

Using this and assumption (3.3.18) we obtain that the function $\boldsymbol{u}$ takes values in $B(|\boldsymbol{u}_0|+c_1)$ if $t\leqslant T$. We calculate the second derivative of γ_i (for $i=1,\dots,n$):

$$\frac{d^2}{dt^2}\gamma_i(\tau,\boldsymbol{x}) = \frac{d}{dt}f_i(\boldsymbol{u}(t,\boldsymbol{\gamma}(t,\boldsymbol{x})))\Big|_{t=\tau} = \sum_{j=1}^m \frac{\partial f_i}{\partial u_j}g_j(\tau,\boldsymbol{\gamma}(\tau,\boldsymbol{x})) = \{(\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{f})\boldsymbol{g}\}_i\,.$$

From this formula we obtain

$$\left|\tfrac{d^2}{dt^2}\gamma_i(\tau,\boldsymbol{x})\right| \leqslant \|\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{f}\|_{B(|\boldsymbol{u}_0|+c_1)}\max_{i=1\dots m}\sup_{\boldsymbol{x}\in\mathbb{R}^n}|g_i(\tau,\boldsymbol{x})|.$$

We multiply this by $T-\tau$ and using (3.3.25) we obtain

$$\|q_i\| \leqslant \|\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{f}\|_{B(|\boldsymbol{u}_0|+c_1)}\int_0^T (T-\tau)\max_{i=1\dots m}\sup_{\boldsymbol{x}\in\mathbb{R}^n}|g_i(\tau,\boldsymbol{x})|\,d\tau\,.$$

Using the inequality $\|\boldsymbol{q}\|\leqslant\sqrt{n}\max q_i$ we arrive at (3.3.26). □

Consider the linearization of the map $\boldsymbol{p}$ at the point $\boldsymbol{x}^\star$ (we recall that $\boldsymbol{x}^\star$ is the point where the leading non zero I_k is negative):

$$p_i(\boldsymbol{x}) = p_i(\boldsymbol{x}^\star) + \sum_{j=1}^n \frac{\partial p_i}{\partial x_j}(\boldsymbol{x}^\star)\triangle x_j + \phi_i(\boldsymbol{\triangle x})\,, \tag{3.3.27}$$

where $\boldsymbol{\triangle x} = \boldsymbol{x}-\boldsymbol{x}^\star$. The remainder term $\phi_i(\boldsymbol{\triangle x})$ can be written as

$$\phi_i(\boldsymbol{\triangle x}) = \frac{1}{2}\sum_{j,k=1}^n \frac{\partial^2 p_i}{\partial x_j\partial x_k}(\boldsymbol{x}^\star+\theta_i\boldsymbol{\triangle x})\triangle x_j\triangle x_k$$

for some $\theta_i\in(0,1)$. Using the definition of the map $\boldsymbol{p}$ (3.3.24) we have

$$|\phi_i(\boldsymbol{\triangle x})| \leqslant \frac{1}{2}|\tilde{\boldsymbol{f}}|_2 T\big(|\triangle x_1|+\dots+|\triangle x_n|\big)^2 \leqslant \frac{n}{2}|\tilde{\boldsymbol{f}}|_2 T\|\boldsymbol{\triangle x}\|^2\,.$$

From this the following inequality for Euclidean norm follows:

$$\|\boldsymbol{\phi}(\boldsymbol{\triangle x})\| \leqslant \sqrt{n}\frac{n}{2}|\tilde{\boldsymbol{f}}|_2 T\|\boldsymbol{\triangle x}\|^2\,. \tag{3.3.28}$$

We need to investigate the matrix

$$A = \tfrac{\partial p_i}{\partial x_j}\Big|_{\boldsymbol{x}=\boldsymbol{x}^\star} = \delta_{ij} + T\tfrac{\partial \tilde{f}_i}{\partial x_j}\Big|_{\boldsymbol{x}=\boldsymbol{x}^\star}. \tag{3.3.29}$$

This $n\times n$ matrix is actually the linear part of the right hand side of (3.3.27).

Proposition 2. *The determinant of the matrix A is negative and bounded away from zero:*

$$det\, A \leqslant -2^{n-1} l^{l/2} (T|\tilde{\boldsymbol{f}}|_1)^{l-1}. \tag{3.3.30}$$

Proof. The determinant of this matrix is expressed by polynomial (3.3.6): $det\, A = P(T)$. Using (3.3.16) we have

$$det\, A \leqslant 1 + \tbinom{n}{1} 1^{1/2} T|\tilde{\boldsymbol{f}}|_1 + \cdots + \tbinom{n}{l-1}(l-1)^{(l-1)/2}(T|\tilde{\boldsymbol{f}}|_1)^{l-1} - \varepsilon T^l \overset{(*)}{\leqslant}$$

$$\tfrac{1}{2} 2^n l^{l/2} (T|\tilde{\boldsymbol{f}}|_1)^{l-1} - \varepsilon T^l = T^{l-1}\big(\tfrac{1}{2} 2^n l^{l/2} |\tilde{\boldsymbol{f}}|_1^{l-1} - \varepsilon T\big).$$

Using (3.3.17) we arrive at (3.3.30). We need to explain inequality $(*)$. For $l = 1$ it follows from the trivial inequality $1 \leqslant 2^{n-1}$. For $l \geqslant 2$ we use the simple fact $(l-1)^{(l-1)/2} \leqslant \frac{1}{2} l^{l/2}$ and inequality (3.3.22) to get

$$1 + \tbinom{n}{1} 1^{1/2} T|\tilde{\boldsymbol{f}}|_1 + \cdots + \tbinom{n}{l-1}(l-1)^{(l-1)/2}(T|\tilde{\boldsymbol{f}}|_1)^{l-1} \leqslant$$
$$\Big(\tbinom{n}{0} + \tbinom{n}{1} + \cdots + \tbinom{n}{l-1}\Big)(l-1)^{(l-1)/2}(T|\tilde{\boldsymbol{f}}|_1)^{l-1} \leqslant 2^n \tfrac{1}{2} l^{l/2} (T|\tilde{\boldsymbol{f}}|_1)^{l-1}.$$

□

Let $\tilde{\mu}_1 \leqslant \tilde{\mu}_2 \leqslant \cdots \leqslant \tilde{\mu}_n$ be the eigenvalues of the matrix $A^t A$. They are real and positive since the matrix $A^t A$ is symmetric and positive definite. Let $\mu_i = \sqrt{\tilde{\mu}_i}$.

Proposition 3. *The stretching coefficient of the linear operator A is at least μ_1, i.e., for any $\vartriangle \boldsymbol{x}$, we have*

$$\|A\vartriangle \boldsymbol{x}\| \geqslant \mu_1 \|\vartriangle \boldsymbol{x}\|\,.$$

Proof. Without loss of generality we can assume that matrix $A^t A$ is diagonal. (A symmetric matrix can be transformed to the diagonal form with the help of an orthogonal change of the coordinates.) Now:

$$\|A\vartriangle \boldsymbol{x}\|^2 = (A\vartriangle \boldsymbol{x}, A\vartriangle \boldsymbol{x}) = (\vartriangle \boldsymbol{x}, A^t A \vartriangle \boldsymbol{x}) = \sum_{i=1}^{n} \tilde{\mu}_i (\vartriangle x_i)^2 \geqslant \tilde{\mu}_1 \|\vartriangle \boldsymbol{x}\|^2\,. \qquad \square$$

Proposition 4. *For the constant μ_1 in the previous proposition, we have:*

$$\mu_1 \geqslant \frac{l^{l/2}}{n^{\frac{3n-1}{4}} (T|\tilde{\boldsymbol{f}}|_1)^{n-l}}\,. \tag{3.3.31}$$

Proof. Let $a = \max\limits_{i,j=1\ldots n} |A_{ij}|$, where A_{ij} are the entries of the matrix A. First we show that

$$\mu_1 \geqslant \frac{|det\, A|}{n^{\frac{3n-1}{4}} a^{n-1}}. \tag{3.3.32}$$

The modulus of any entry of the matrix $A^t A$ is no greater than na^2. The modulus of any $(n-1)\times(n-1)$ minor of the matrix $A^t A$ is no greater than the volume of a $(n-1)$-dimensional parallelepiped with sides of length less than or equal to $\sqrt{n-1}na^2$. Using the representations of the inverse matrix via minors (the Cramer's rule) we see that the modulus of any entry of the matrix $(A^t A)^{-1}$ is not bigger than $\frac{(\sqrt{n-1}na^2)^{n-1}}{det\, A^t A}$. Applying the Gersgorin theorem (see [24]) to the matrix $(A^t A)^{-1}$ we see that the eigenvalues of the matrix $(A^t A)^{-1}$ are no greater than

$$n\,\frac{(\sqrt{n-1}na^2)^{n-1}}{det\, A^t A} \leqslant \frac{n^{\frac{n-1}{2}} n^n a^{2(n-1)}}{|det\, A|^2}$$

and (3.3.32) follows.

From (3.3.29) if follows that for the entries of the matrix A we have $|A_{ij}| \leqslant \delta_{ij} + |\tilde{\boldsymbol{f}}|_1 T$. Using (3.3.22) we obtain $a \leqslant 2|\tilde{\boldsymbol{f}}|_1 T$. Using (3.3.30) we arrive at (3.3.31). □

Consider the sphere $S_r(\boldsymbol{x}^\star)$ (w.r.t. Euclidean norm) with the centre at the point $\boldsymbol{x}^\star$ and with the radius:

$$r = \frac{\mu_1}{\sqrt{n} n |\tilde{\boldsymbol{f}}|_2 T}\,. \tag{3.3.33}$$

Suppose that (3.3.19) does not hold. Then from (3.3.26) and (3.3.31) it follows that

$$\|\boldsymbol{q}\| < \tfrac{1}{2}\mu_1 r\,.$$

Besides this, from (3.3.28) it follows that

$$\|\boldsymbol{\phi}(\vartriangle \boldsymbol{x})\| \leqslant \tfrac{1}{2}\mu_1 r\,.$$

Let $\boldsymbol{h}(\vartriangle \boldsymbol{x}) = \boldsymbol{\phi}(\vartriangle \boldsymbol{x}) + \boldsymbol{q}(\boldsymbol{x}^\star + \vartriangle \boldsymbol{x})$. Then due to (3.3.23) and (3.3.27) we obtain:

$$\boldsymbol{\gamma}(T, \boldsymbol{x}^\star + \vartriangle \boldsymbol{x}) = \boldsymbol{p}(\boldsymbol{x}^\star) + A\vartriangle \boldsymbol{x} + \boldsymbol{h}(\vartriangle \boldsymbol{x})\,. \tag{3.3.34}$$

We recall that $\boldsymbol{x}^\star + \vartriangle \boldsymbol{x} = \boldsymbol{x}$. Using the inequality $\|\boldsymbol{h}(\vartriangle \boldsymbol{x})\| < \mu_1 r$ and proposition 3, we have

$$\|A\vartriangle \boldsymbol{x}\| > \|\boldsymbol{h}(\vartriangle \boldsymbol{x})\| \qquad \text{for } \|\vartriangle \boldsymbol{x}\| = r \text{ , i.e., } \boldsymbol{x} \in S_r(\boldsymbol{x}^\star)\,. \tag{3.3.35}$$

From this inequality it follows that the Gauss spherical map $\Gamma : S_r(\boldsymbol{x}^\star) \to S_1(\boldsymbol{0})$

$$_\triangle\boldsymbol{x} \mapsto \frac{A\,_\triangle\boldsymbol{x} + \boldsymbol{h}(_\triangle\boldsymbol{x})}{\|A\,_\triangle\boldsymbol{x} + \boldsymbol{h}(_\triangle\boldsymbol{x})\|} \tag{3.3.36}$$

is well defined. From (3.3.35) it also follows that this map is homotopic to the map

$$_\triangle\boldsymbol{x} \mapsto \frac{A\,_\triangle\boldsymbol{x}}{\|A\,_\triangle\boldsymbol{x}\|}\,.$$

Hence the degrees of these maps coincide and are equal to $\operatorname{sign} det\, A = -1$ (see [7], II, chpt. 3). On the other hand, the map (3.3.36) can be written as

$$_\triangle\boldsymbol{x} \mapsto \frac{\boldsymbol{\gamma}(\boldsymbol{x}^\star + {}_\triangle\boldsymbol{x}) - \boldsymbol{p}(\boldsymbol{x}^\star)}{\|\boldsymbol{\gamma}(\boldsymbol{x}^\star + {}_\triangle\boldsymbol{x}) - \boldsymbol{p}(\boldsymbol{x}^\star)\|}\,.$$

The degree of this map is equal to

$$\sum_{\substack{\boldsymbol{y}\in\boldsymbol{\gamma}^{-1}(\boldsymbol{p}(\boldsymbol{x}^\star))\\ \boldsymbol{y} \text{ in the ball}}} \operatorname{sign} det \left(\frac{\partial \boldsymbol{\gamma}(T,\boldsymbol{\xi})}{\partial \boldsymbol{\xi}}\Big|_{\boldsymbol{\xi}=\boldsymbol{y}} \right) \geqslant 0$$

(see [7], II, chpt. 3). This contradiction proves Theorem 3.3.4. □

Corollary. *Let $\boldsymbol{u}(0,\boldsymbol{x})$ be periodic with compact fundamental periodic domain $\mathbb{T}$. It follows that $\int_{\mathbb{T}} I_k(\boldsymbol{x})d\boldsymbol{x} = 0$ (see lemma 3.3.1); so our theorem is applicable for the periodic case iff not all $I_k(\boldsymbol{x})$ are identically zero. In this case we can put $l = \max\{k \in 1,2,\dots,n : I_k \not\equiv 0\}$ and $\varepsilon = -\min I_l(\boldsymbol{x})$.*

Using the inequalities

$$\max_{i=1\dots m} \int_0^T \sup|g_i(\tau,\cdot)|\,d\tau \geqslant \frac{1}{m}\int_0^T \sum_{i=1}^m \sup|g_i(\tau,\cdot)|\,d\tau \geqslant$$

$$\frac{1}{m}\int_0^T \max_{i=1\dots m}\sup|g_i(\tau,\cdot)|\,d\tau \geqslant \frac{1}{Tm}\int_0^T (T-\tau)\max_{i=1\dots m}\sup|g_i(\tau,\cdot)|\,d\tau$$

we obtain

$$\max_{i=1\dots m}\int_0^T \sup|g_i(\tau,\cdot)|\,d\tau \geqslant \sup_{c_1>0}\min\left\{c_1, \tfrac{c_2(c_1)}{Tm}\right\}. \tag{3.3.37}$$

3.4 Fourier coefficients

In this Section we outline some information concerning the behaviour of the Fourier coefficients of solutions for equation (3.1.1) that can be obtained from our previous

results. It turns out that this behaviour is consistent with the so-called Kolmogorov – Obukhov asymptotics. This fact allows us to formulate the Kolmogorov – Obukhov law for solutions of the generalised Burgers equation (3.1.1) and specify bounds for the spectral parameters of this hypothesis.

What is the Kolmogorov – Obukhov spectral law? It is actually a hypothesis. It can be derived from a theory of turbulence developed by Kolmogorov in 1941 see [8], Chpt 6. This theory, usually called K41 theory is based on three hypotheses concerning the behaviour of a turbulent flow. The mathematical justification of this theory is an open problem, and only few rigorous results have been obtained in this direction.

The Kolmogorov – Obukhov spectral law describes the distribution of the Fourier coefficients $\widehat{\boldsymbol{v}}_{\boldsymbol{k}}(t) = \frac{1}{\ell^{n/2}}\int \boldsymbol{v}(t,\boldsymbol{r})e^{\frac{-i2\pi\boldsymbol{k}\boldsymbol{r}}{\ell}}d\boldsymbol{r}$ of a velocity field $\boldsymbol{v}(t,\boldsymbol{r})$ that is a turbulent solution to the Navier Stokes system with viscosity ν. The distribution obtained has the following structure: There exist non-negative constants $\gamma_1 \leqslant \gamma_2$ and $\varkappa$ such that for $\left(\frac{1}{\nu}\right)^{\gamma_1} < |\boldsymbol{k}| < \left(\frac{1}{\nu}\right)^{\gamma_2}$ (the inertial range) we have $\langle|\widehat{\boldsymbol{v}}_{\boldsymbol{k}}|^2\rangle \sim \left(\frac{1}{|\boldsymbol{k}|}\right)^{\varkappa}$. Here $\langle\cdot\rangle$ denotes averaging over time and over a band of wave vectors. For $|\boldsymbol{k}| > \left(\frac{1}{\nu}\right)^{\gamma_2}$ (the dissipation range) the quantities $|\widehat{\boldsymbol{v}}_{\boldsymbol{k}}|$ decay faster than any power of $\frac{1}{|\boldsymbol{k}|}$. The theory does not say much about the energy range $|\boldsymbol{k}| < \left(\frac{1}{\nu}\right)^{\gamma_1}$.

The quantity ν^{γ_2} is named the Kolmogorov dissipation scale and $\varkappa$ is named the exponent of the Kolmogorov – Obukhov spectral law.

We consider solutions of the generalised Burgers equation (3.1.1) rather than solutions of the Navier Stokes system.

Assuming that the Kolmogorov – Obukhov type hypothesis is valid for solutions of the generalised Burgers equation (3.1.1), we derive that both the inertial and the dissipation ranges are non empty and find estimates for the Kolmogorov dissipation scale ($\nu^{\gamma_2} \in [\nu, \nu^{1/2}]$) and estimates for the exponent of the spectral law ($\varkappa \in [1,2]$).

In the 1D case (see Chapter 2) it is shown that the Kolmogorov dissipation scale is equal to ν and the exponent of the spectral law is equal to 2.

We start with an auxiliary lemma. This lemma is of a general nature and can, for example, be applied to the results obtained in [1] and [18]. In this lemma and

subsequently, $|\boldsymbol{s}|$ stands for the Euclidean norm of the integer vector $\boldsymbol{s} \in \mathbb{Z}^n$ and $\nu \in (0, \nu_0)$ is a small positive parameter.

Lemma. *Suppose that there exist positive numbers γ_1, γ_2, q_1, q_2, k_0, ν_0 such that the positive quantities $\hat{a}^2_{\boldsymbol{s}}(\nu)$, satisfy to the following inequalities:*

$$\sum_{\boldsymbol{s}\in\mathbb{Z}^n} |\boldsymbol{s}|^{2k} \hat{a}^2_{\boldsymbol{s}} \geqslant c_k \left(\tfrac{1}{\nu}\right)^{2\gamma_1 k - 2q_1}, \quad \text{for } k \geqslant k_0, \tag{3.4.1}$$

$$\sum_{\boldsymbol{s}\in\mathbb{Z}^n} |\boldsymbol{s}|^{2k} \hat{a}^2_{\boldsymbol{s}} \leqslant C_k \left(\tfrac{1}{\nu}\right)^{2\gamma_2 k - 2q_2}, \quad \text{for } k \geqslant 1, \tag{3.4.2}$$

$$\sum_{\boldsymbol{s}\in\mathbb{Z}^n} \hat{a}^2_{\boldsymbol{s}} \leqslant C_0 \tag{3.4.3}$$

for apropriate positive constants C_k and c_k. Then for any $k \geqslant k_0$, $\varepsilon > 0$, and $\nu < \nu(k, \varepsilon)$ we have

$$\sum_{(\frac{1}{\nu})^{\gamma_1 - \frac{q_1}{k} - \varepsilon} < |\boldsymbol{s}| < (\frac{1}{\nu})^{\gamma_2 + \varepsilon}} \hat{a}^2_{\boldsymbol{s}} \geqslant \tfrac{1}{2} c_k \nu^{2k(\gamma_2 - \gamma_1 + \varepsilon) + 2q_1}, \tag{3.4.4}$$

and for any $M \geqslant 1$, any positive real numbers z and ε, and any $\nu < \nu(M, z, \varepsilon)$ we have

$$\sum_{(\frac{1}{\nu})^{z-\varepsilon} < |\boldsymbol{s}| < (\frac{1}{\nu})^{z+\varepsilon}} \hat{a}^2_{\boldsymbol{s}} \leqslant C_M \nu^{2M(z - \gamma_2 - \varepsilon) + 2q_2}. \tag{3.4.5}$$

Remark. We also note that due to (3.4.3), the last sum is less than or equal to C_0.

Proof. In what follows y is a positive real number. Using (3.4.3) and (3.4.2), we get upper bounds for low and high Fourier modes:

$$\sum_{|\boldsymbol{s}| < \nu^{-y}} |\boldsymbol{s}|^{2k} \hat{a}^2_{\boldsymbol{s}} \leqslant \left(\tfrac{1}{\nu}\right)^{2ky} \sum \hat{a}^2_{\boldsymbol{s}} \leqslant C_0 \left(\tfrac{1}{\nu}\right)^{2ky}. \tag{3.4.6}$$

If $y < \gamma_1 - \frac{q_1}{k}$, then for small ν the term on the right is much less than $\left(\frac{1}{\nu}\right)^{2\gamma_1 k - 2q_1}$. Next, for any positive integer M, we have

$$\sum_{|\boldsymbol{s}| > \nu^{-y}} |\boldsymbol{s}|^{2k} \hat{a}^2_{\boldsymbol{s}} \leqslant \nu^{2My} \sum |\boldsymbol{s}|^{2M+2k} \hat{a}^2_{\boldsymbol{s}} \leqslant C_{k+M} \left(\tfrac{1}{\nu}\right)^{2M(\gamma_2 - y) + 2\gamma_2 k - 2q_2}. \tag{3.4.7}$$

If $y > \gamma_2 + \frac{(\gamma_2 - \gamma_1)k + q_1 - q_2}{M}$, then for small ν the term on the right is much less than $\left(\frac{1}{\nu}\right)^{2\gamma_1 k - 2q_1}$. Combining inequalities (3.4.6), (3.4.7) with sufficiently large M, and

(3.4.1), we see that for small enough ν:

$$\sum_{(\frac{1}{\nu})^{\gamma_1 - \frac{q_1}{k} - \varepsilon} < |\boldsymbol{s}| < (\frac{1}{\nu})^{\gamma_2 + \varepsilon}} \hat{a}^2_{\boldsymbol{s}} \geqslant \nu^{2k(\gamma_2 + \varepsilon)} \sum_{(\frac{1}{\nu})^{\gamma_1 - \frac{q_1}{k} - \varepsilon} < |\boldsymbol{s}| < (\frac{1}{\nu})^{\gamma_2 + \varepsilon}} |\boldsymbol{s}|^{2k} \hat{a}^2_{\boldsymbol{s}} \geqslant \tfrac{1}{2} \nu^{2k(\gamma_2 + \varepsilon)} \sum_{\boldsymbol{s}\in\mathbb{Z}^n} |\boldsymbol{s}|^{2k} \hat{a}^2_{\boldsymbol{s}}.$$

Using (3.4.1) again, we arrive at (3.4.4). Using (3.4.7) with $k = 0$ and $y = z - \varepsilon$, we arrive at (3.4.5). □

We now specify the parameters of this lemma for the situation we are considering in this Chapter.

For any multi-index β such that $|\beta| \leqslant k$ we have:

$$\left\|D^\beta \boldsymbol{u}\right\|^2_{k-|\beta|} = \sum_{|\gamma| = k - |\beta|} \tbinom{|\gamma|}{\gamma} \left|D^{\gamma+\beta}\boldsymbol{u}\right|^2_{L_2} \leqslant \sum_{|\gamma| = k - |\beta|} \tbinom{|\gamma+\beta|}{\gamma+\beta} \left|D^{\gamma+\beta}\boldsymbol{u}\right|^2_{L_2} \leqslant \sum_{|\alpha| = k} \tbinom{|\alpha|}{\alpha} \left|D^\alpha \boldsymbol{u}\right|^2_{L_2} = \|\boldsymbol{u}\|^2_k. \tag{3.4.8}$$

Lemma 3.4.1. *Let v be a function on the torus $\mathbb{T}^n$ with zero mean value. Then with $q = \left[\frac{n}{2}\right] + 1$ we have*

$$\|v\|_q \geqslant K_n \left|v\right|_{L_\infty}. \tag{3.4.9}$$

Proof. Consider the Fourier expansion of v

$$v(\boldsymbol{x}) = \sum_{\boldsymbol{s}\in\mathbb{Z}^n} \hat{v}_{\boldsymbol{s}} e_{\boldsymbol{s}}(\boldsymbol{x}), \quad \text{where} \quad \hat{v}_{\boldsymbol{s}} = \int_{\mathbb{T}^n} v(\boldsymbol{x}) \overline{e_{\boldsymbol{s}}(\boldsymbol{x})} d\boldsymbol{x}.$$

Here $e_{\boldsymbol{s}}(\boldsymbol{x}) = \frac{1}{\ell^{n/2}} \exp\left(\frac{2\pi i \boldsymbol{s}\boldsymbol{x}}{\ell}\right)$ is the orthonormal basis on $L_2(\mathbb{T}^n)$. Then we have

$$\left|v\right|_{L_\infty} \leqslant \tfrac{1}{\ell^{n/2}} \sum_{\boldsymbol{s}\in\mathbb{Z}^n} |\hat{v}_{\boldsymbol{s}}|$$

and

$$\|v\|^2_q = \left(\tfrac{2\pi}{\ell}\right)^{2q} \sum_{\boldsymbol{s}\in\mathbb{Z}^n} |\boldsymbol{s}|^{2q} |\hat{v}_{\boldsymbol{s}}|^2.$$

Since $\hat{v}_0 = 0$ we have

$$\sum_{\boldsymbol{s}\in\mathbb{Z}^n} |\hat{v}_{\boldsymbol{s}}| = \sum_{\boldsymbol{s}\in\mathbb{Z}^n\setminus\{0\}} \tfrac{1}{|\boldsymbol{s}|^q} |\boldsymbol{s}|^q |\hat{v}_{\boldsymbol{s}}| \leqslant \Big(\sum_{\boldsymbol{s}\in\mathbb{Z}^n\setminus\{0\}} \tfrac{1}{|\boldsymbol{s}|^{2q}}\Big)^{\frac{1}{2}} \Big(\sum_{\boldsymbol{s}\in\mathbb{Z}^n} |\boldsymbol{s}|^{2q} |\hat{v}_{\boldsymbol{s}}|^2\Big)^{\frac{1}{2}}.$$

We arrive at (3.4.9) with the constant $K_n = \dfrac{\ell^{n/2}\left(\frac{2\pi}{\ell}\right)^{2q}}{\left(\sum\limits_{\boldsymbol{s}\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{|\boldsymbol{s}|^{2q}}\right)^{\frac{1}{2}}}$. □

From inequalities (3.4.8) and (3.4.9) it follows that for any $k \geqslant q+1$ and any index j we have

$$\|\boldsymbol{u}\|_k^2 \geqslant \left\|\tfrac{\partial^{k-q}\boldsymbol{u}}{\partial x_j^{k-q}}\right\|_q^2 \geqslant K_n^2 \left|\tfrac{\partial^{k-q}\boldsymbol{u}}{\partial x_j^{k-q}}\right|_{L_\infty}^2 .$$

Theorem 3.4.2. *For any $k \geqslant 2+q$ and for any $\nu > 0$, we have*

$$\frac{1}{T}\int_0^T \|\boldsymbol{u}\|_k^2 dt \geqslant K_n^2 r_{k-q}^2 \left(\tfrac{1}{\nu}\right)^{k-q} . \tag{3.4.10}$$

Proof. Using Theorem 3.1.2 we obtain:

$$\frac{1}{T}\int_0^T \|\boldsymbol{u}\|_k^2 dt \geqslant K_n^2 \frac{1}{T}\int_0^T \left|\tfrac{\partial^{k-q}\boldsymbol{u}}{\partial x_j^{k-q}}\right|_{L_\infty}^2 dt \geqslant K_n^2 \left(\frac{1}{T}\int_0^T \left|\tfrac{\partial^{k-q}\boldsymbol{u}}{\partial x_j^{k-q}}\right|_{L_\infty} dt\right)^2 \geqslant K_n^2 \frac{r_{k-q}^2}{\nu^{k-q}} .$$

□

On the other hand, from (3.1.3) it follows that for any $k \geqslant 0$, for sufficiently small ν, we have

$$\frac{1}{T}\int_0^T \|\boldsymbol{u}\|_k^2 dt \leqslant \widetilde{C}_k \left(\tfrac{1}{\nu}\right)^{2k}. \tag{3.4.11}$$

Consider the orthonormal basis on $L_2(\mathbb{T}^n)$ formed by the exponentials $e_{\boldsymbol{s}}(\cdot)$, $\boldsymbol{s} \in \mathbb{Z}^n$, where

$$e_{\boldsymbol{s}}(\boldsymbol{x}) = \tfrac{1}{\ell^{n/2}} \exp\left(\tfrac{2\pi i \boldsymbol{s}\boldsymbol{x}}{\ell}\right) .$$

The Fourier expansion of $\boldsymbol{u}(\boldsymbol{x})$ has the form

$$\boldsymbol{u}(t,\boldsymbol{x}) = \sum_{\boldsymbol{s}\in\mathbb{Z}^n} \hat{\boldsymbol{u}}_{\boldsymbol{s}}(t) e_{\boldsymbol{s}}(\boldsymbol{x}) , \quad \text{where} \quad \hat{\boldsymbol{u}}_{\boldsymbol{s}}(t) = \int_{\mathbb{T}^n} \boldsymbol{u}(t,\boldsymbol{x})\overline{e_{\boldsymbol{s}}(\boldsymbol{x})} d\boldsymbol{x} .$$

The Sobolev quasinorm $\|\cdot\|_k$ is given by $\|\boldsymbol{u}\|_k^2 = \left(\frac{2\pi}{\ell}\right)^{2k} \sum_{\boldsymbol{s}\in\mathbb{Z}^n} |\boldsymbol{s}|^{2k} |\hat{\boldsymbol{u}}_{\boldsymbol{s}}|^2$, where $|\hat{\boldsymbol{u}}_{\boldsymbol{s}}|$ denotes m-dimensional Hermitian norm of the vector $\hat{\boldsymbol{u}}_{\boldsymbol{s}}$. Let

$$\hat{a}_{\boldsymbol{s}}^2 = \frac{1}{T}\int_0^T |\hat{\boldsymbol{u}}_{\boldsymbol{s}}(t)|^2 dt,$$

then

$$\sum_{\boldsymbol{s}\in\mathbb{Z}^n} |\boldsymbol{s}|^{2k} \hat{a}_{\boldsymbol{s}}^2 = \left(\frac{\ell}{2\pi}\right)^{2k} \frac{1}{T}\int_0^T \|\boldsymbol{u}\|_k^2 \, dt.$$

Now we can apply the Lemma on p.68 to the quantities $\hat{a}_{\boldsymbol{s}}^2$ above with the parameters:

$$\gamma_1 = \tfrac{1}{2} \qquad q_1 = \tfrac{q}{2} \qquad c_k = \left(\tfrac{\ell}{2\pi}\right)^{2k} K_n^2 r_{k-q}^2 \qquad k_0 = 2+q$$
$$\gamma_2 = 1 \qquad q_2 = 0 \qquad C_k = \left(\tfrac{\ell}{2\pi}\right)^{2k} \widetilde{C}_k$$

Indeed, with this settings inequalities (3.4.1) – (3.4.3) are equivalent to (3.4.10), (3.4.11). We recall that $q = \left[\frac{n}{2}\right] + 1$.

This lemma imply that for any $k \geqslant q+2$, $\varepsilon > 0$ and $\nu < \nu(k,\varepsilon)$ we have:

$$\sum_{(\frac{1}{\nu})^{\frac{1}{2}-\frac{q}{2k}-\varepsilon} < |\boldsymbol{s}| < (\frac{1}{\nu})^{1+\varepsilon}} \hat{a}_{\boldsymbol{s}}^2 \geqslant \tfrac{1}{2} c_k \nu^{k+2k\varepsilon+q},$$

and for any $k \geqslant 0$, any positive real numbers z and ε, and any $\nu < \nu(k,z,\varepsilon)$ we have

$$\sum_{(\frac{1}{\nu})^{z-\varepsilon} < |\boldsymbol{s}| < (\frac{1}{\nu})^{z+\varepsilon}} \hat{a}_{\boldsymbol{s}}^2 \leqslant C_k \nu^{2k(z-1-\varepsilon)}.$$

From these inequalities we obtain the following conditional result:

If the quantities $\hat{a}_{\boldsymbol{s}}^2(\nu)$ (which are averaged Fourier coefficients) obeys a Kolmogorov – Obukhov type spectral law then the Kolmogorov dissipation scale lies between $\nu^{1/2}$ and ν and the power of the spectral law is in the interval $[1,2]$.

3.5 Low bounds for spatial derivatives of solutions of Navier Stokes System

The Navier Stokes equation can be regarded as a special case of the equation (3.1.1) when $m = n$, $\boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{u}$ and $\boldsymbol{h}$ represent the gradient of the pressure term. This allows us to extend the results of this chapter to the solutions of Navier-Stokes system.

In this section we consider the dynamics of a vector field $\boldsymbol{u} = \boldsymbol{u}(t,\boldsymbol{x})$ on the torus $\mathbb{T}^n = {}^{\mathbb{R}^n}\!/_{\ell\mathbb{Z}^n}$ described by the Navier Stokes system:

$$\partial_t \boldsymbol{u} + \nabla_{\boldsymbol{u}} \boldsymbol{u} = \nu\Delta\boldsymbol{u} + \nabla p(t,\boldsymbol{x}) , \tag{3.5.1}$$
$$\operatorname{div} \boldsymbol{u} = 0, \tag{3.5.2}$$

with positive viscosity ν. We assume that the initial state $\boldsymbol{u}(0,\boldsymbol{x}) = \boldsymbol{u}_0$ is C^2-smooth. It is known that the strong solution of the system (3.5.1),(3.5.2) exists for $0 \leqslant t < T_0 \leqslant \infty$, and $|\boldsymbol{u}(t,\cdot)|_{L_\infty} \to \infty$ as $t \to T_0$ if $T_0 < \infty$. In the paper [17] it is shown that $T_0 \geqslant \frac{C(n)\,\ell^2}{\nu R^2(1+\max\{0,\log R\})^2}$, where $R = \frac{\ell}{\nu}|\boldsymbol{u}_0|_{L_\infty}$ is the Reynolds number.

Remark. Since the mean value of the $\boldsymbol{u}$ is constant, we have that spatial derivatives of the flow $\boldsymbol{u}$ blow up with $|\boldsymbol{u}(t,\cdot)|_{L_\infty}$. Due to this reason we assume that $|D^\alpha \boldsymbol{u}(t,\cdot)|_{L_\infty} = \infty$ for $t \geqslant T_0$ and any multiindex α.

Theorem 3.5.1. *Suppose that the initial state $\boldsymbol{u}_0$ is non-degenerate vector field. Then there exist a ν-independent positive real number T and ν-independent positive real numbers $\varkappa_2$, $\varkappa_3$, $\varkappa_4$, ... such that for any $\nu > 0$ and for any $k \geqslant 2$ we have:*

$$\max\Big\{ \max_{\substack{j=1\dots n\\ i=1\dots m}} \frac{1}{T}\int_0^T \sup_{\boldsymbol{x}\in\mathbb{T}^n} \Big|\tfrac{\partial^k u_i}{\partial {x_j}^k}(t,\boldsymbol{x})\Big|\, dt\,,\ \max_{j=1\dots n} \frac{1}{T}\int_0^T \tfrac{1}{\nu^{k/2}} \sup_{\boldsymbol{x}\in\mathbb{T}^n} \big|\tfrac{\partial^{k-1} p}{\partial x_j^{k-1}}(t,\boldsymbol{x})\big| dt \Big\} \geqslant \frac{\varkappa_k}{\nu^{k/2}}\,. \tag{3.5.3}$$

As the next result shows, the assumption of non-degeneracy can not be removed. To show this we can restrict ourself to the 2D case, since any solution of the 2D Navier Stokes system generate a solution of a higher-dimensional system by adding zeroth components of the field $\boldsymbol{u}$ and fictitious variables. The non-degeneracy is preserved by this operation. We remark that in 2D case the non-degeneracy is a necessary and sufficient condition for Theorem 3.5.1.

Lemma 3.5.2. *Let $\boldsymbol{u}$ be the solution of 2D Navier Stokes system with a degenerate initial state. Then for any $t > 0$ and $k \geqslant 0$ we have* $\max|\boldsymbol{u}(t,\boldsymbol{x})|_{C^k(\mathbb{T}^2)} \leqslant \max|\boldsymbol{u}(0,\boldsymbol{x})|_{C^k(\mathbb{T}^2)}$ *and* $\nabla p \equiv 0$.

Proof. In the 2D case any degenerate initial state has the form:

$$\boldsymbol{u}_0(\boldsymbol{x}) = \tbinom{b_2}{-b_1}\varphi_0(b_1x_1 + b_2x_2) + \tbinom{c_1}{c_2} \tag{3.5.4}$$

for a suitable function $\varphi_0(\cdot)$ and real numbers b_1, b_2, c_1, and c_2. (See Theorem 3.3.2). In this case, the solution of the Cauchy problem for (3.5.1), (3.5.2) remains of the form (3.5.4):

$$\boldsymbol{u}(t,\boldsymbol{x}) = \tbinom{b_2}{-b_1}\varphi(t, b_1x_1 + b_2x_2) + \tbinom{c_1}{c_2}\,,$$

where the function φ satisfies to the following linear parabolic equation with constant coefficients:

$$\varphi_t + (b_1c_1 - b_2c_2)\varphi' = (b_1^2 + b_2^2)\nu\varphi''\,.$$

From this it follows that the derivatives of $\boldsymbol{u}$ are decreasing and $\nabla p \equiv 0$. □

We note that in the 2D case class of periodic solutions with degenerate initial state coincides with the class of periodic solutions with $\nabla p \equiv 0$. Indeed, if the initial state is degenerate, then $\nabla p \equiv 0$ due to Lemma 3.5.2. To show the converse we apply the identity

$$\Delta p(t,x) = -2I_2(t,x), \tag{3.5.5}$$

where $I_k(t,x)$ are the coefficients of the characteristic polynomial of the matrix $\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}(t,\boldsymbol{x})$:

$$\det\big(\tfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} - \lambda\mathbf{1}\big) = (-\lambda)^n + (-\lambda)^{n-1}I_1(t,x) + \dots + I_n(t,x).$$

The degeneracy of the initial state is equivalent to the condition $I_1(0,\cdot) \equiv I_2(0,\cdot) \equiv \dots \equiv I_n(0,\cdot) \equiv 0$. The coefficient $I_1 \equiv 0$ due to (3.5.2).

The identity (3.5.5) is also valid in higher dimensions.

Proof of the Theorem. As we will show, the assertion of the theorem holds for any $\boldsymbol{u}$ and p, that satisfy (3.5.1). We, actually, do not need use (3.5.2). First, from Theorem 3.1.2 we find $T = T(\boldsymbol{u})$. If $T_0 < T$ then the l.h.s. of (3.5.3) is equal to infinity due to made remark. Suppose $T < T_0$, then from Theorem 3.1.2 again we find $c = c(\boldsymbol{u}_0)$ and $r_k = r_k(\boldsymbol{u}_0)$ $(k \geqslant 2)$ such that if

$$\max_{j=1\dots n} \frac{1}{T}\int_0^T \sup_{\boldsymbol{x}\in\mathbb{T}^n} \big|\tfrac{\partial p}{\partial x_j}(t,\boldsymbol{x})\big| dt < \tfrac{c}{2} \tag{3.5.6}$$

then for any $k \geqslant 2$ we have

$$\max_{\substack{j=1\dots n\\ i=1\dots m}} \frac{1}{T}\int_0^T \sup_{\boldsymbol{x}\in\mathbb{T}^n} \Big|\tfrac{\partial^k u_i}{\partial {x_j}^k}(t,\boldsymbol{x})\Big|\, dt \geqslant \tfrac{r_k}{\nu^{k/2}}\,. \tag{3.5.7}$$

If

$$\max_{j=1\dots n} \frac{1}{T}\int_0^T \sup_{\boldsymbol{x}\in\mathbb{T}^n} \big|\tfrac{\partial p}{\partial x_j}(t,\boldsymbol{x})\big| dt \geqslant \tfrac{c}{2}$$

then for any $k \geqslant 2$ we have:

$$\max_{j=1\dots n} \frac{1}{T}\int_0^T \sup_{\boldsymbol{x}\in\mathbb{T}^n} \big|\tfrac{\partial^{k-1} p}{\partial x_j^{k-1}}(t,\boldsymbol{x})\big| dt \geqslant \tfrac{c}{2\ell^{k-1}}$$

Inequality (3.5.3) is proved with the constant $\varkappa_k = \min\{r_k, \frac{c}{2\ell^{k-1}}\}$. □

Chapter 4

On Non-linear Schrödinger equations

4.1 Introduction

In this Section we study solutions of the equation

$$\dot{u} = e^{i\theta}\nu\Delta u + i\,|u|^{2p}\,u\,. \tag{4.1.1}$$

Here $\nu > 0$ is a small positive real number (viscosity), by $e^{i\theta}$ we denote a complex number on the unit circle in $\mathbb{C}$ (θ is an arbitrary fixed real number), p is a positive integer and by $u = u(t, \boldsymbol{x})$ we denote a complex valued function on $[0,\infty) \times K^n$, which satisfies (4.1.1). Here K^n is a domain in $\mathbb{R}^n$. Usually this is the cube $[0,\ell]^n$.

In [18] upper and low estimates for the derivatives of solutions of (4.1.1) were studied. In particular the following result was obtained:

Theorem 4.1.1. (Theorem 3 in [18].) *Suppose that u is a smooth solution of (4.1.1) with $p > 1$ on $[0,\infty) \times K^n$, with a smooth initial state u_0. Here $K^n = [0,1]^n$. No boundary conditions for u on ∂K^n are assumed. Suppose, that*

$$\operatorname{osc}_{K^n} |u_0(x)| \stackrel{def}{=} \sup_{x,y\in K^n} |u_0(x)| - |u_0(y)| \geqslant 1\,.$$

Then for any $\kappa < \frac{1}{3}$ and $m \geqslant 2$ there exists a positive constant $\nu' = \nu'(\kappa, m, |u_0|_{L_\infty})$

and $T_0 = T_0(u_0)$, $0 \leqslant T_0 \leqslant \nu^{-\frac{1}{3}}$ such that

$$|u(T_0)|_{C^m(K^n)} \geqslant \nu^{-m\kappa} \quad \forall \nu < \nu'$$

and $\operatorname{osc}_{K^n} |u(T_0,\cdot)| \geqslant \frac{1}{2}$.

In this section we first show that κ above can be replaced with $\frac{1}{3}$, i.e., $\nu^{-m\kappa}$ above can be replaced with $C_m\nu^{-\frac{1}{3}m}$. Explicit values for the constants

$$C_m = C_m(\min|u_0|, \max|u_0|)$$

and ν' can be easily derived from our results. We will present explicit constant for the lower estimates in a slightly different form. We then illustrate how to get lower estimates for locally time-averaging of Sobolev seminorms when it is known that classical derivative is big at some point. We actually follow paper [18], simplifying some proofs. These lower estimates for locally time-averaging Sobolev seminorms do not follow from the previous theorem alone. Some extra arguments and information about the function u should be used. In our case we again use that the function u satisfies equation (4.1.1) and hence some inequalities for its Sobolev norms can be obtained using Gagliardo-Nirenberg type inequalities. For this approach we should assume that $\Re\{e^{i\theta}\} \geqslant 0$.

The second part is a generalisation of the first one. It is self-contained and can be read independently from the first subsection. We start the second subsection by proving a lower estimate for the local time average of the second classical seminorm (i.e., supremum norm of second derivatives) over the time-interval $[0,T]$, where $T = c_t\nu^{-\frac{1}{3}}$. There we obtain the same power of the viscosity ($\nu^{-2\frac{1}{3}}$) as in the first subsection, where the lower estimate is obtained at some unknown time $t^\star \in [0,T]$. The lower estimates for locally time-averaged Sobolev seminorms are a trivial consequence of this. In fact the final result obtained is stronger than that obtained in the first subsection. Also we do not make the assumption $\Re\{e^{i\theta}\} \geqslant 0$ and do not use Gagliardo-Nirenberg type inequalities here.

In the third part we obtain explicit constants for upper estimates of Sobolev seminorms for periodic solutions of equation (4.1.1) with $\theta = 0$. In particularly, we obtain

the following inequality

$$\|u(t,\cdot)\|_k \leqslant \tfrac{\|u_0\|_0}{\nu^{k/2}} \max\Big\{ \big(\tfrac{k}{t}\big)^{k/2},\, (2C_{k,p,n})^{\frac{k}{2}} |u_0|_{L_\infty}^{pk} \Big\},$$

where $C_{k,p,n} = 4^{k^2}(2p+1)^k n^k$.

We do not claim that any of constants obtained are optimal. However it is usually the case that inequalities of this type are expressed in terms of an arbitrarily large constant C. Here we have been able to improve this by finding particular values for the constants.

4.2 Lower estimates I

Theorem 4.2.1. *Let $u = u(t,\boldsymbol{x})$ be a smooth complex valued function, which satisfies equation (4.1.1) with $p > 1$ in the interior of the domain $[0,\infty)\times[0,\ell]^n$.*

Suppose that there exists a point $\boldsymbol{x}_1 \in [0,\ell]^n$ such that

$$0 < |u(0,\boldsymbol{x}_1)| \stackrel{\text{def}}{=} a < \sup_{\boldsymbol{x}\in[0,\ell]^n} |u(0,\boldsymbol{x})| \stackrel{\text{def}}{=} b\,. \tag{4.2.1}$$

Let $\varepsilon > 0$ be a real number such that the ε-neighbourhoods of the numbers a and b do not intersect and do not contain zero, i.e., $\varepsilon < a$ and $\varepsilon < \frac{b-a}{2}$.

Suppose that the positive real numbers c_d and c_t satisfy the following inequalities:

$$nc_t c_d^2 \leqslant \varepsilon, \tag{4.2.2}$$

$$c_d < c_t \frac{\big((b-\varepsilon)^{2p} - (a+\varepsilon)^{2p}\big)(a-\varepsilon)}{4n\ell(b+\varepsilon)^{1/2}}, \tag{4.2.3}$$

and that ν is small enough to satisfy the following two inequalities:

$$c_d^2 \nu^{-2/3}\ell^2 \geqslant 4(b+\varepsilon)\,, \tag{4.2.4}$$

$$c_t\nu^{-1/3}\big((b-\varepsilon)^{2p} - (a+\varepsilon)^{2p}\big) \geqslant \frac{4n\ell(b+\varepsilon)^{1/2}c_d}{a-\varepsilon}\nu^{-1/3} + \frac{\varepsilon}{b-\varepsilon} + \frac{\varepsilon}{a-\varepsilon}\,. \tag{4.2.5}$$

Then there exists a time $t^\star \in [0,T]$, where $T = c_t\nu^{-1/3}$, such that

$$\max_{j=1,\dots,n} \sup_{\boldsymbol{x}\in[0,\ell]^n} \left|\frac{\partial^2 u}{\partial x_j^2}(t^\star,\boldsymbol{x})\right| \geqslant c_d^2\nu^{-2/3}, \tag{4.2.6}$$

$$\sup_{\substack{\boldsymbol{x}\in[0,\ell]^n\\ t\in[0,t^\star]}} \big|u(t,\boldsymbol{x})\big| \leqslant b+\varepsilon\,, \tag{4.2.7}$$

$$\inf_{t\in[0,t^\star]} \operatorname{osc} |u(t,\cdot)| \geqslant b-a-2\varepsilon\,. \tag{4.2.8}$$

Proof. Let $\boldsymbol{x}_2$ be a point at which the supremum in (4.2.1) is achieved. For simplicity we assume that $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are internal points of the cube $[0,\ell]^n$. The general case is obtained by a limiting process. Also we assume that for all points $\boldsymbol{x} \in (\boldsymbol{x}_1,\boldsymbol{x}_2)$ we have $|u(0,\boldsymbol{x})| \in (a,b)$ $\big($otherwise we can consider a subinterval of $(\boldsymbol{x}_1,\boldsymbol{x}_2)\big)$.

Suppose that (4.2.6) does not hold, i.e.:

$$\left|\frac{\partial^2 u}{\partial x_j^2}(t,\boldsymbol{x})\right| < c_d^2\nu^{-2/3} \quad \text{for all } j = 1,\dots,n,\ \ t\in[0,T],\ \ \boldsymbol{x}\in[0,\ell]^n. \tag{4.2.9}$$

Then from the equation (4.1.1) it follows that for $t \in [0,T]$ the function $|u|\,(t,\boldsymbol{x})$ is Lipschitz in t with the Lipschitz constant $nc_d^2\nu^{1/3}$. Using this and inequality (4.2.2), we have

$$\big||u(0,\boldsymbol{x})| - |u(t,\boldsymbol{x})|\big| < \varepsilon \tag{4.2.10}$$

for all $\boldsymbol{x} \in [0,\ell]^n$ and $t \in [0,T]$. By virtue of condition (4.2.4) we can apply (A.2.2) to get

$$\left|\tfrac{\partial u}{\partial x_j}\right| \leqslant 2(b+\varepsilon)^{1/2}c_d\nu^{-1/3} \quad \text{for all } j = 1,\dots,n,\ t\in[0,T],\ \boldsymbol{x}\in[0,\ell]^n. \tag{4.2.11}$$

Hence the images of the intervals $\big[(0,\boldsymbol{x}_1),(0,\boldsymbol{x}_2)\big]$ and $\big[(T,\boldsymbol{x}_1),(T,\boldsymbol{x}_2)\big]$ are curves each of length no greater than

$$n\ell \max\left|\tfrac{\partial u}{\partial x_j}\right| \leqslant 2n\ell(b+\varepsilon)^{1/2}c_d\nu^{-1/3}\,.$$

By virtue of (4.2.10) the image of the rectangle with the vertices $(0,\boldsymbol{x}_1)$, $(0,\boldsymbol{x}_2)$, $(T,\boldsymbol{x}_2)$ and $(T,\boldsymbol{x}_1)$ under the mapping u belongs to the ring $a-\varepsilon \leqslant |u| \leqslant b+\varepsilon$ and is contractible. We denote the boundary of this rectangle by Γ. Consider the map $\tilde\varphi : \Gamma \to S^1\,, \quad \tilde\varphi(t,\boldsymbol{x}) = \frac{|u(t,\boldsymbol{x})|}{u(t,\boldsymbol{x})}$, i.e., $\tilde\varphi(t,\boldsymbol{x}) = \arg u(t,\boldsymbol{x})$. We represent S^1 as $S^1 = {}^{\mathbb{R}}\!/_{2\pi\mathbb{Z}}$. Let $\varphi : \Gamma \to \mathbb{R}$ be the lifting of this map to the real line $\mathbb{R}$. Since the map $\tilde\varphi$ is contractible, this lifting exists and is defined up to a multiple of 2π. On the "vertical" sides of the rectangle Γ (i.e., on the sides parallel to the time axis) we have the inequality.

$$u^{2p} - \tfrac{|\nu\Delta u|}{u} \leqslant \tfrac{\partial\varphi}{\partial t} \leqslant u^{2p} + \tfrac{|\nu\Delta u|}{u}\,. \tag{4.2.12}$$

For the side $\big[(0,\boldsymbol{x}_2),(T,\boldsymbol{x}_2)\big]$ we assign the number $r_0=|\varphi(T,\boldsymbol{x}_2)-\varphi(0,\boldsymbol{x}_2)|$. Similarly for the other three sides of the rectangle Γ we define the numbers r_1, r_2 and r_3. Using the triangle inequality we have:

$$T\left((b-\varepsilon)^{2p}-\tfrac{\nu n c_d^2\nu^{-2/3}}{b-\varepsilon}\right)\leqslant r_0\leqslant r_1+r_2+r_3 \\ \leqslant T\left((a+\varepsilon)^{2p}+\tfrac{\nu n c_d^2\nu^{-2/3}}{a-\varepsilon}\right)+2\tfrac{2n\ell(b+\varepsilon)^{1/2}c_d\nu^{-1/3}}{a-\varepsilon}. \quad (4.2.13)$$

Accounting for (4.2.2), this inequality contradicts inequality (4.2.5). We set $t^\star$ to be the first moment of time when inequality (4.2.6) holds, then inequality (4.2.10) is satisfied for $\boldsymbol{x}\in[0,\ell]^n$ and $t\in[0,t^\star]$, and we arrive at (4.2.7) and (4.2.8). □

We say that a function u satisfy odd-periodic boundary conditions if for any $j=1,\dots,n$ we have:

$$u(t,\boldsymbol{x})=u(t,x_1,\dots,x_j+2\ell,\dots,x_n)=-u(t,x_1,\dots,-x_j,\dots,x_n)\,. \quad (4.2.14)$$

Theorem 4.2.2. *Let u be an odd periodic solution of equation (4.1.1). Let*

$$c_d=\tfrac{(4^{2p}-3^{2p})^{1/3}}{2n^{2/3}\ell^{1/3}5^{2p/3+1/2}6^{1/6}}|u_0|^{2p/3+1/2} \quad \textit{and} \quad c_t=\tfrac{4(6n\ell^2 5^{4p})^{1/3}}{(4^{2p}-3^{2p})^{2/3}|u_0|^{4p/3}}. \quad (4.2.15)$$

Suppose that

$$\nu<\tfrac{(4^{2p}-3^{2p})\ell^2}{2304n^2 5^{2p}}|u_0|^{2p}; \quad (4.2.16)$$

then there exists $t^\star\in[0,T]$, where $T=c_t\nu^{-1/3}$ such that

$$\max_{j=1,\dots,n}\sup_{\boldsymbol{x}\in[0,\ell]^n}\left|\frac{\partial^2 u}{\partial x_j^2}(t^\star,\boldsymbol{x})\right|\geqslant c_d^2\nu^{-2/3}, \quad (4.2.17)$$

$$\sup_{\substack{\boldsymbol{x}\in[0,\ell]^n\\ t\in[0,t^\star]}}\big|u(t^\star,\boldsymbol{x})\big|\leqslant\frac{6}{5}|u_0|\,. \quad (4.2.18)$$

Proof. By virtue of the odd-periodicity, the magnitude $|u_0(\boldsymbol{x})|$ takes all values between zero and $\max|u_0|$. We set $b=|u_0|$, $a=\frac{2}{5}|u_0|$, $\varepsilon=\frac{1}{5}|u_0|$ and use Theorem 4.2.1. □

Remark. We can see from the proof that Theorem 4.2.2 holds not only for odd-periodic solutions, but more generally for any periodic solution that takes the value zero when $t=0$.

On the space of 2ℓ-periodic functions

$$u(t,\boldsymbol{x})=u(t,x_1,\dots,x_j+2\ell,\dots,x_n) \quad (4.2.19)$$

we consider the family of (semi-) norms

$$\|u\|_k^2=\int_{[0,2\ell]^n}u\,(-\Delta)^k\bar{u}\,d^n\boldsymbol{x}=\sum_{|\alpha|=k}\tbinom{|\alpha|}{\alpha}\big|D^\alpha u\big|^2_{L_2([0,2\ell]^n)}\,. \quad (4.2.20)$$

Here $\binom{|\alpha|}{\alpha}=\binom{|\alpha|}{\alpha_1,\dots,\alpha_n}=\frac{(\alpha_1+\dots+\alpha_n)!}{\alpha_1!\cdots\alpha_n!}$. These are the coefficients in the generalised binomial equation:

$$(x_1+x_2+\ldots+x_n)^k=\sum_{|\alpha|=k}\tbinom{k}{\alpha}\boldsymbol{x}^\alpha\,.$$

We note that $\sum_{|\alpha|=k}\binom{k}{\alpha}=n^k$.

Any smooth function u from the space (4.2.14) can be expanded as a sine series:

$$u(t,\boldsymbol{x})=\sum_{\boldsymbol{s}\in\mathbb{N}^n}a_{\boldsymbol{s}}(t)\tfrac{1}{\ell^{n/2}}\prod_{j=1}^n\sin\left(\tfrac{\pi}{\ell}s_jx_j\right)\,, \quad (4.2.21)$$

where $a_{\boldsymbol{s}}(t)=\frac{1}{\ell^{n/2}}\int_{[0,2\ell]^n}u(t,\boldsymbol{x})\prod_{j=1}^n\sin\left(\frac{\pi}{\ell}s_jx_j\right)dx$. The factor $\frac{1}{\ell^{n/2}}$ is included in order to compensate for the $L_2([0,2\ell]^n)$ norm of each product of sines.

The norms (4.2.20) for the space (4.2.14) can be written as

$$\|u(t,\cdot)\|_k^2=\sum_{\boldsymbol{s}\in\mathbb{N}^n}\tfrac{\pi^{2k}}{\ell^{2k}}\boldsymbol{s}^{2k}|a_{\boldsymbol{s}}(t)|^2. \quad (4.2.22)$$

Here $\boldsymbol{s}^2=s_1^2+\ldots+s_n^2$.

Lemma. *Let u be a 2ℓ-periodic solution of equation (4.1.1). Then for any time t we have:*

$$\frac{1}{2}\frac{d}{dt}\|u\|_k^2\leqslant-\nu e^{i\theta}{}_{Re}\|u\|_{k+1}^2+C_{k,p,n}|u|_{L_\infty}^{2p}\|u\|_k^2. \quad (4.2.23)$$

where $C_{k,p,n}=4^{k^2}(2p+1)^kn^k$.

Proof. We multiply the equation (4.1.1) by $(-\Delta)^k\bar{u}$, integrate over the period $[0,2\ell]^n$ and take the real part to get

$$\frac{1}{2}\frac{d}{dt}\|u\|_k^2=-\nu e^{i\theta}{}_{Re}\|u\|_{k+1}^2+\Re\int_{[0,2\ell]^n}i|u|^{2p}u(-\Delta)^k\bar{u}\,d^nx\,. \quad (4.2.24)$$

For the sake of brevity, we will omit the domain of integration $[0,2\ell]^n$ from the integral sign. To complete proof of the Lemma, we need to prove the inequality

$$\Big|\int |u|^{2p}u(-\Delta)^k\bar{u}\,d^nx\Big| \leqslant 4^{k^2}(2p+1)^k n^k |u|^{2p}_{L_\infty}\int u(-\Delta)^k\bar{u}dx. \qquad (4.2.25)$$

Integrating the l.h.s. of this by parts k times, we obtain

$$\int |u|^{2p}u\sum\tfrac{\partial^2}{\partial x_{j_1}^2}\cdots\tfrac{\partial^2}{\partial x_{j_k}^2}\bar{u}dx = (-1)^k\int\sum\tfrac{\partial}{\partial x_{j_1}}\cdots\tfrac{\partial}{\partial x_{j_k}}(u^p\bar{u}^pu)\tfrac{\partial}{\partial x_{j_1}}\cdots\tfrac{\partial}{\partial x_{j_k}}\bar{u}dx. \quad (4.2.26)$$

Both the r.h.s. and the l.h.s. contain n^k terms. Now we use the formula for differentiating a product:

$$\tfrac{\partial}{\partial x_{j_1}}\cdots\tfrac{\partial}{\partial x_{j_k}}(v_1v_2\cdots v_{2p+1}) = \sum_{\beta_1+\cdots+\beta_{2p+1}=(1,\ldots,1)}\prod_{l=1}^{2p+1}\left(\tfrac{\partial}{\partial x_{j_1}},\ldots,\tfrac{\partial}{\partial x_{j_k}}\right)^{\beta_l}v_l.$$

Here β_l are k-dimensional multi-indexes and the dimension of $(1,\ldots,1)$ is also k. The sum in the r.h.s. contains $(2p+1)^k$ terms. Hence the r.h.s. of the (4.2.26) can be represented as a sum of $(2p+1)^kn^k$ terms, each of them of the form

$$\int D^{\alpha_1}v_1\cdots D^{\alpha_{2p+1}}v_{2p+1}D^{\alpha}v_{2p+2}\,dx, \qquad (4.2.27)$$

where $|\alpha|=k$ and $\alpha_1+\cdots+\alpha_{2p+1}=\alpha$; $v_l=u$ for odd l and $v_l=\bar{u}$ for even l. Using the Hölder inequality we estimate the absolute value of the (4.2.27) as:

$$\Big|\int D^{\alpha_1}v_1\cdots D^{\alpha_{2p+1}}v_{2p+1}D^{\alpha}v_{2p+2}dx\Big| \leqslant \Big(\prod_{l=1}^{2p+1}|D^{\alpha_l}v_l|_{L_{\frac{2k}{|\alpha_l|}}}\Big)\,|D^{\alpha}v_{2p+2}|_{L_2} = \mathfrak{A}\,.$$

Using the Gagliardo-Nirenberg inequality (see [13], pp. 106-107)

$$|D^{\alpha_l}v_l|_{L_{\frac{2k}{|\alpha_l|}}} \leqslant 4^{|\alpha_l|(k-|\alpha_l|)}|v_l|_{L_\infty}^{1-\frac{|\alpha_l|}{k}}\|v_l\|_k^{\frac{|\alpha_l|}{k}}\,,$$

we obtain

$$\mathfrak{A} \leqslant \Big(\prod_{l=1}^{2p+1}4^{|\alpha_l|(k-|\alpha_l|)}|v_l|_{L_\infty}^{1-\frac{|\alpha_l|}{k}}\|v_l\|_k^{\frac{|\alpha_l|}{k}}\Big)\|v_{2p+2}\|_k \leqslant 4^{k^2}|u|^{2p}_{L_\infty}\|u\|_k^2\,.$$

To obtain the last inequality we used the simple inequality $|\alpha_l|(k-|\alpha_l|)\leqslant|\alpha_l|k$ and the fact that $\sum|\alpha_l|=k$. We arrive at (4.2.25), so the Lemma is proved. □

Extrapolating (4.2.17) and (4.2.18), we obtain for any $k\geqslant 2$:

$$\max_{j=1,\ldots,n}\sup_{\boldsymbol{x}\in[0,\ell]^n}\left|\frac{\partial^k u}{\partial x_j^k}(t^\star,\boldsymbol{x})\right| \geqslant \frac{1}{\sqrt{3}}\frac{c_d^k\nu^{-k/3}}{\left(\frac{6}{5}\,|u_0|_{L_\infty}\right)^{\frac{k-2}{2}}}\,. \qquad (4.2.28)$$

The value $t^\star$ comes from Theorem 4.2.2. To get (4.2.28) we used the interpolation inequality $|f^{(l)}|_{L_\infty(\mathbb{R})}\leqslant C_{l,k}|f|^{1-l/k}_{L_\infty(\mathbb{R})}|f^{(k)}|^{l/k}_{L_\infty(\mathbb{R})}$ (see [14]) with $l=2$ and

$$f(x) = u(t^\star,\tilde{x}_1,\ldots,\tilde{x}_{j-1},x,\tilde{x}_{j+1},\ldots,\tilde{x}_n),$$

where j is the index of the derivative and $(\tilde{x}_1,\ldots,\tilde{x}_n)$ is the point where supremum in (4.2.17) is achieved. We also used the inequality $C_{2,k}^{k/2}\leqslant\sqrt{3}$ which is proved in the Appendix.

Using the inequalities (3.4.8) and (3.4.9) we derive from (4.2.28) that for any $k\geqslant 2+q$, where $q=\left[\frac{n}{2}\right]+1$, we have

$$\|u(t^\star,\cdot)\|_k^2 \geqslant \big\|\tfrac{\partial^{k-q}u}{\partial x_j^{k-q}}\big\|_q^2 \geqslant K_n^2\big|\tfrac{\partial^{k-q}u}{\partial x_j^{k-q}}\big|^2_{L_\infty} \geqslant c_k''\left(\tfrac{1}{\nu}\right)^{\frac{2}{3}(k-q)},$$

where c_k'' are ν-independent constants.

Suppose, that $e^{i\theta}{}_{Re}\geqslant 0$; then using Theorem 4.2.2 we derive from (4.2.23) the following inequality:

$$\tfrac{d}{dt}\|u\|_k^2\leqslant c'\|u\|_k^2 \quad \text{ for all } \ t\in[0,t^\star],$$

where $c'=2C_{k,p,n}\left(\frac{6}{5}\,|u_0|_{L_\infty}\right)^{2p}$. From this it follows that

$$\int_0^T\|u\|_k^2dt \geqslant \int_0^{t^\star}\|u\|_k^2dt \geqslant \tfrac{1}{c'}\int_0^{t^\star}\tfrac{d}{dt}\|u\|_k^2dt \geqslant \tfrac{c_k''}{c'}\left(\tfrac{1}{\nu}\right)^{\frac{2}{3}(k-q)} - \tfrac{1}{c'}\|u_0\|_k^2.$$

The final result is:

Suppose $k\geqslant 2+q$; then for small enough ν we have

$$\frac{1}{T}\int_0^T\|u\|_k^2dt \geqslant c\left(\tfrac{1}{\nu}\right)^{\frac{2}{3}(k-q)-\frac{1}{3}}\,.$$

Here $T=c_t\nu^{1/3}$; c_t and c are ν-independent constants.

Actually, the constant c_t can be chosen arbitrarily, e.g., we could chose $c_t=1$, but c and the required smallness of the viscosity ν depend on this choice, see Theorem 4.2.1.

4.3 Lower estimates II

Theorem 4.3.1. *Let $u = u(t, \boldsymbol{x})$ be a smooth complex valued function, which satisfies equation (4.1.1) with $p > 1$ in the interior of the domain $[0, \infty) \times [0, \ell]^n$.*

Suppose that there exists a point $\boldsymbol{x}_1 \in [0, \ell]^n$ such that (4.2.1) holds. Let $\varepsilon > 0$ be as in Theorem 4.2.1 of the previous subsection, i.e., a real number such that the ε-neighbourhoods of the numbers a and b do not intersect and do not contain zero, i.e., $\varepsilon < a$ and $\varepsilon < \frac{b-a}{2}$. (The values a and b are defined in (4.2.1)).

Suppose that the positive real numbers c_d and c_t satisfy inequalities (4.2.2) and

$$c_d < c_t \frac{\big((b-\varepsilon)^{2p} - (a+\varepsilon)^{2p}\big)(a-\varepsilon)}{4(2n)^{3/2}\ell(b+\varepsilon)^{1/2}}, \tag{4.2.3$'$}$$

Assume that ν is small enough to satisfy the following two inequalities:

$$2nc_d^2\nu^{-2/3}\ell^2 \geqslant 4(b+\varepsilon)\,, \tag{4.2.4$'$}$$

$$c_t\nu^{-1/3}\big((b-\varepsilon)^{2p} - (a+\varepsilon)^{2p}\big) \geqslant \frac{4(2n)^{3/2}\ell(b+\varepsilon)^{1/2}c_d}{a-\varepsilon}\nu^{-1/3} + \frac{2\varepsilon}{b-\varepsilon} + \frac{2\varepsilon}{a-\varepsilon}\,. \tag{4.2.5$'$}$$

Then there exists a time $t^\star \in [0, T]$, where $T = c_t\nu^{-1/3}$, such that

$$\max_{j=1,\dots,n} \int_0^{t^\star} \sup_{\boldsymbol{x}\in[0,\ell]^n} \left|\tfrac{\partial^2 u}{\partial x_j^2}(t,\boldsymbol{x})\right| dt \geqslant c_t c_d^2 \nu^{-1}, \tag{4.2.6$'$}$$

$$\sup_{\substack{\boldsymbol{x}\in[0,\ell]^n \\ t\in[0,t^\star]}} \big|u(t,\boldsymbol{x})\big| \leqslant b+\varepsilon\,, \tag{4.2.7$'$}$$

$$\inf_{t\in[0,t^\star]} \operatorname{osc}|u(t,\cdot)| \geqslant b-a-2\varepsilon\,. \tag{4.2.8$'$}$$

Proof. Let $\boldsymbol{x}_2$ be a point at which the supremum in (4.2.1) is achieved. For simplicity we assume that $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are internal points of the cube $[0, \ell]^n$. The general case is obtained by a limiting process. Also we assume that for all points $\boldsymbol{x} \in (\boldsymbol{x}_1, \boldsymbol{x}_2)$ we have $|u(0, \boldsymbol{x})| \in (a, b)$ $\big($otherwise we can consider a subinterval of $(\boldsymbol{x}_1, \boldsymbol{x}_2)\big)$.

Suppose that (4.2.6$'$) does not hold, i.e.:

$$\int_0^T \sup_{\boldsymbol{x}\in[0,\ell]^n} \left|\tfrac{\partial^2 u}{\partial x_j^2}(t,\boldsymbol{x})\right| < c_t c_d^2 \nu^{-1} \quad \text{for all } j = 1, \dots, n\,. \tag{4.2.9$'$}$$

From equation (4.1.1) it follows that

$$\big||u(0,\boldsymbol{x})| - |u(t,\boldsymbol{x})|\big| \leqslant \int_0^t |\nu\Delta u(t,\boldsymbol{x})|dt\,.$$

Combining the last two inequalities and (4.2.2), we get

$$\big||u(0,\boldsymbol{x})| - |u(t,\boldsymbol{x})|\big| < \varepsilon \tag{4.2.10$'$}$$

for all $\boldsymbol{x} \in [0, \ell]^n$ and $t \in [0, T]$. Summing up the inequalities (4.2.9$'$) we get

$$\int_0^T \sum_{j=1}^n \sup_{\boldsymbol{x}\in[0,\ell]^n} \left|\tfrac{\partial^2 u}{\partial x_j^2}(t,\boldsymbol{x})\right| dt < nc_t c_d^2 \nu^{-1}\,.$$

From this it follows that there exists $\hat{t} \in [\frac{1}{2}T, T]$ such that

$$\sum_{j=1}^n \sup_{\boldsymbol{x}\in[0,\ell]^n} \left|\tfrac{\partial^2 u}{\partial x_j^2}(\hat{t},\boldsymbol{x})\right| < 2nc_d^2\nu^{-\frac{2}{3}}\,.$$

Hence

$$\left|\tfrac{\partial^2 u}{\partial x_j^2}(\hat{t},\boldsymbol{x})\right| < 2nc_d^2\nu^{-\frac{2}{3}} \qquad \text{for all } j = 1, \dots, n,\ \boldsymbol{x} \in [0,\ell]^n\,.$$

By virtue of condition (4.2.4$'$) we can apply (A.2.2) to get

$$\left|\tfrac{\partial u(\hat{t},\boldsymbol{x})}{\partial x_j}\right| \leqslant 2(b+\varepsilon)^{1/2}\sqrt{2n}c_d\nu^{-1/3} \quad \text{for all } j = 1, \dots, n,\ \boldsymbol{x} \in [0,\ell]^n\,.$$

Hence the images of the intervals $\big[(0, \boldsymbol{x}_1), (0, \boldsymbol{x}_2)\big]$ and $\big[(\hat{t}, \boldsymbol{x}_1), (\hat{t}, \boldsymbol{x}_2)\big]$ are curves each of length no greater than

$$n\ell \max\left|\tfrac{\partial u}{\partial x_j}\right| \leqslant (2n)^{3/2}\ell(b+\varepsilon)^{1/2}c_d\nu^{-1/3}\,.$$

By virtue of (4.2.10$'$) the image of the rectangle with the vertices $(0, \boldsymbol{x}_1)$, $(0, \boldsymbol{x}_2)$, $(\hat{t}, \boldsymbol{x}_2)$ and $(\hat{t}, \boldsymbol{x}_1)$ under the mapping u belongs to the ring $a - \varepsilon \leqslant |u| \leqslant b + \varepsilon$ and is contractible. We denote the boundary of this rectangle by Γ. Consider the map $\tilde{\varphi} : \Gamma \to S^1$, $\quad \tilde{\varphi}(t, \boldsymbol{x}) = \frac{|u(t,\boldsymbol{x})|}{u(t,\boldsymbol{x})}$, i.e., $\tilde{\varphi}(t, \boldsymbol{x}) = \arg u(t, \boldsymbol{x})$. We represent S^1 as $S^1 = {}^{\mathbb{R}}\!/_{2\pi\mathbb{Z}}$. Let $\varphi : \Gamma \to \mathbb{R}$ be the lifting of this map to the real line $\mathbb{R}$. Since the map $\tilde{\varphi}$ is contractible, this lifting exists and is defined up to a multiple of 2π. On the "vertical" sides of the rectangle Γ (i.e., on the sides parallel to the time axis) we have the inequality.

$$u^{2p} - \tfrac{|\nu\Delta u|}{u} \leqslant \tfrac{\partial\varphi}{\partial t} \leqslant u^{2p} + \tfrac{|\nu\Delta u|}{u}\,. \tag{4.2.12$'$}$$

For the side $\big[(0, \boldsymbol{x}_2), (\hat{t}, \boldsymbol{x}_2)\big]$ we assign the number $r_0 = |\varphi(\hat{t}, \boldsymbol{x}_2) - \varphi(0, \boldsymbol{x}_2)|$. Similarly for the other three sides of the rectangle Γ we define the numbers r_1, r_2 and r_3.

Integrating (4.2.12′) and using the triangle inequality we have:

$$\hat{t}\left((b-\varepsilon)^{2p}-\tfrac{\nu n c_d^2\nu^{-2/3}}{b-\varepsilon}\right)\leqslant r_0\leqslant r_1+r_2+r_3$$
$$\leqslant\hat{t}\left((a+\varepsilon)^{2p}+\tfrac{\nu n c_d^2\nu^{-2/3}}{a-\varepsilon}\right)+2\tfrac{(2n)^{3/2}\ell(b+\varepsilon)^{1/2}c_d\nu^{-1/3}}{a-\varepsilon}.$$

Accounting for (4.2.2), this inequality contradicts inequality (4.2.5′), since $\hat{t}>\frac{1}{2}c_t\nu^{-\frac{1}{3}}$. We set $t^\star$ to be the first moment of time when inequality (4.2.6′) holds, then inequality (4.2.10′) is satisfied for $\boldsymbol{x}\in[0,\ell]^n$ and $t\in[0,t^\star]$, and we arrive at (4.2.7′) and (4.2.8′). □

Now we turn to the 2ℓ-periodic case.

Theorem 4.3.2. *Let $u=u(t,\boldsymbol{x})$ be a 2ℓ-periodic solution of equation (4.1.1) with $p>1$. Under the assumptions of Theorem 4.3.1, for any $k\geqslant 2$, and for any $t\in[t^\star,T]$, we have*

$$\max_{j=1,\dots,n}\frac{1}{t}\int_0^t\sup\left|\tfrac{\partial^k u}{\partial x_j^k}(\tau,\cdot)\right|d\tau\geqslant\tfrac{{c_d}^k}{\sqrt{3}(b+\varepsilon)^{\frac{k-2}{2}}}\nu^{-\frac{k}{3}},\tag{4.3.1}$$

where $T=c_t\nu^{-\frac{1}{3}}$ and $t^\star\in[0,T]$ are given by Theorem 4.3.1.

Proof. Take the index j for which the maximum in (4.2.6′) is achieved. In the following chain of inequalities $|\cdot|$ denotes the $L_\infty(\mathbb{T}^n)$-norm, where $\mathbb{T}^n$ is the periodicity box. We have

$$\frac{1}{t}\int_0^t\left|\tfrac{\partial^k u}{\partial x_j^k}(\tau,\cdot)\right|d\tau\geqslant\frac{1}{t}\int_0^{t^\star}\left|\tfrac{\partial^k u}{\partial x_j^k}(\tau,\cdot)\right|d\tau\geqslant\frac{t^\star}{t}\frac{1}{t^\star}\int_0^{t^\star}\frac{\left|\frac{\partial^2 u}{\partial x_j^2}(\tau,\cdot)\right|^{k/2}}{\sqrt{3}\sup|u(\tau,\cdot)|^{\frac{k-2}{2}}}d\tau\geqslant$$
$$\frac{t^\star}{t}\frac{\left(\frac{1}{t^\star}\int_0^{t^\star}\left|\frac{\partial^2 u}{\partial x_j^2}\right|d\tau\right)^{k/2}}{\sqrt{3}(b+\varepsilon)^{\frac{k-2}{2}}}\geqslant\frac{1}{t(t^\star)^{\frac{k}{2}-1}}\frac{\left(c_t c_d^2\nu^{-1}\right)^{k/2}}{\sqrt{3}(b+\varepsilon)^{\frac{k-2}{2}}}\geqslant\frac{{c_d}^k\nu^{-\frac{k}{3}}}{\sqrt{3}(b+\varepsilon)^{\frac{k-2}{2}}}.$$

To obtain the last inequality we have used the inequalities $t\leqslant c_t\nu^{-\frac{1}{3}}$ and $t^\star\leqslant c_t\nu^{-\frac{1}{3}}$. The Theorem is proved. □

For any multi-index β such that $|\beta|\leqslant k$ we have:

$$\left\|D^\beta u\right\|_{k-|\beta|}^2=\sum_{|\gamma|=k-|\beta|}\tbinom{|\gamma|}{\gamma}\left|D^{\gamma+\beta}u\right|_{L_2}^2\leqslant$$
$$\sum_{|\gamma|=k-|\beta|}\tbinom{|\gamma+\beta|}{\gamma+\beta}\left|D^{\gamma+\beta}u\right|_{L_2}^2\leqslant\sum_{|\alpha|=k}\tbinom{|\alpha|}{\alpha}\left|D^\alpha u\right|_{L_2}^2=\|u\|_k^2\,.$$

From this and (3.4.9) it follows that for any $k\geqslant 1+q$, where $q=\left[\frac{n}{2}\right]+1$ and for any $j=1,\dots,n$, we have

$$\frac{1}{T}\int_0^T\|u\|_k^2dt\geqslant\frac{1}{T}\int_0^T\left\|\tfrac{\partial^{k-q}u}{\partial x_j^{k-q}}\right\|_q^2dt\geqslant$$
$$K_n^2\frac{1}{T}\int_0^T\left|\tfrac{\partial^{k-q}u}{\partial x_j^{k-q}}\right|_{L_\infty}^2dt\geqslant K_n^2\left(\frac{1}{T}\int_0^T\left|\tfrac{\partial^{k-q}u}{\partial x_j^{k-q}}\right|_{L_\infty}dt\right)^2.\tag{4.3.2}$$

Theorem 4.3.3. *Let $u=u(t,\boldsymbol{x})$ be a 2ℓ-periodic solution of equation (4.1.1) with $p>1$. Under the assumptions of Theorem 4.3.1, for any $k\geqslant 2+q$, where $q=\left[\frac{n}{2}\right]+1$, we have*

$$\frac{1}{T}\int_0^T\|u\|_k^2dt\geqslant\tfrac{K_n^2{c_d}^{k-q}}{3(b+\varepsilon)^{k-q-2}}\left(\tfrac{1}{\nu}\right)^{\frac{2}{3}(k-q)},\tag{4.3.3}$$

where $T=c_t\nu^{-\frac{1}{3}}$. The value K_n comes from Lemma 3.4.1.

Proof. The theorem follows from (4.3.2) and Theorem 4.3.2. □

4.4 Upper estimates

In this subsection we prove an upper estimates for Sobolev seminorms for 2ℓ-periodic solutions of equation (4.1.1) with $\theta=0$. The special case of odd 2ℓ-periodic solutions is considered separately, because for this class of solutions we additionally obtain the exponential decay of seminorms with time.

If u is a periodic solution of equation (4.1.1), then multiplying equation (4.1.1) by $\bar{u}$, integrating over the periodicity-box $[0,2\ell]^n$ and taking the real part, we get

$$\tfrac{1}{2}\tfrac{d}{dt}\|u(t,\cdot)\|_0^2=-\nu\cos(\theta)\|u(t,\cdot)\|_1^2.\tag{4.4.1}$$

In what follows, we assume that $\theta=0$, i.e., equation (4.1.1) takes the form

$$\dot{u}=\nu\Delta u+i|u|^{2p}u\,.\tag{4.4.2}$$

Lemma. *For any 2ℓ-periodic solution of equation (4.4.2), we have*

$$\|u(t,\cdot)\|_0\leqslant\|u_0\|_0\,.\tag{4.4.3}$$

For any odd 2ℓ-periodic solution of equation (4.4.2), we have

$$\|u(t,\cdot)\|_0 \leqslant e^{-\nu\frac{\pi^2}{\ell^2}nt}\,\|u_0\|_0\,. \tag{4.4.4}$$

Proof. From (4.4.1) with $\theta = 0$ we have $\frac{1}{2}\frac{d}{dt}\,\|u(t,\cdot)\|_0^2 \leqslant 0$ and (4.4.3) follows.

From (4.2.22) it follows that for any function u from the of space odd 2ℓ-periodic functions (4.2.14) we have

$$\|u\|_1^2 \geqslant \tfrac{\pi^2}{\ell^2}n\|u\|_0^2\,.$$

Combining this and (4.4.1) with $\theta = 0$, we obtain

$$\tfrac{d}{dt}\,\|u(t,\cdot)\|_0^2 \leqslant -2\nu\tfrac{\pi^2}{\ell^2}n\,\|u(t,\cdot)\|_0^2\,.$$

Hence,

$$\|u(t,\cdot)\|_0^2 \leqslant e^{-2\nu\frac{\pi^2}{\ell^2}nt}\,\|u_0\|_0^2\,.$$

This inequality is equivalent to (4.4.4). □

Lemma. (Maximum Principal) *For any 2ℓ periodic solution of (4.4.2) we have*

$$|u(t,\cdot)|_{L_\infty} \leqslant |u_0|_{L_\infty}\,. \tag{4.4.5}$$

For any odd 2ℓ-periodic solution of (4.4.2) we have

$$|u(t,\cdot)|_{L_\infty} \leqslant |u_0|_{L_\infty} \inf_{\varkappa\in[0,1)} \frac{e^{-nt\nu\frac{\varkappa^2\pi^2}{\ell^2}}}{\cos^n\frac{\pi\varkappa}{2}}\,. \tag{4.4.6}$$

Proof. (In this proof we follow paper [18]). We write $u = re^{i\varphi}$. The function r is a positive continuous function; outside the zero-set $\Sigma = r^{-1}(0)$ it is smooth and satisfies equation

$$\tfrac{\partial}{\partial t}r = \nu\Delta r - \nu r|\nabla\varphi|^2\,.$$

To prove (4.4.6) we consider the region $\Pi = [0,\infty)\times[0,\ell]^n$ and the function

$$\xi(t,\boldsymbol{x}) = |u_0|_{L_\infty}\, e^{-nt\nu\frac{\varkappa^2\pi^2}{\ell^2}} \prod_{j=1}^n \frac{\cos\left(\varkappa\frac{\pi}{\ell}(x_j-\frac{\ell}{2})\right)}{\cos\frac{\pi\varkappa}{2}} \tag{4.4.7}$$

in this region. This function is positive in Π and solves there the equation $\xi_t = \nu\Delta\xi$. Besides, $\xi \geqslant |u_0|_{L_\infty}$ in $\{0\}\times[0,\ell]^n$. Now let us consider the function $h = \xi - r$. In $\Pi\setminus\Sigma$ we have $h_t = \nu\Delta h + \nu r|\nabla\varphi|^2$. Since h is nonnegative on the boundary

$$\partial\left(\Pi\setminus\Sigma\right) = \big([0,\infty)\times\partial[0,\ell]^n\big)\cup\big(\{0\}\times[0,\ell]^n\big)\cup\partial\Sigma$$

($\partial\Sigma$ is the boundary of Σ in Π) and since $\nu r|\nabla\varphi|^2 \geqslant 0$, then by the maximum principle $h \geqslant 0$ in $\Pi\setminus\Sigma$. (We apply the principle for a parabolic equation in an arbitrary domain. In the form we need it can be found e.g. in [22].) It means that everywhere in Π we have:

$$r(t,\boldsymbol{x}) \leqslant \xi(t,\boldsymbol{x}) \leqslant |u_0|_{L_\infty}\, e^{-nt\nu\frac{\varkappa^2\pi^2}{\ell^2}}\cos^{-n}\tfrac{\pi\varkappa}{2}\,.$$

Thus (4.4.6) is proved for odd 2ℓ-periodic solutions of equation (4.4.2).

To prove (4.4.5) for 2ℓ-periodic solutions of equation (4.4.2) we replace $\Pi\setminus\Sigma$ with the periodic domain $[0,\infty)\times\mathbb{T}^n\setminus\Sigma$ and consider the constant function $\xi(t,\boldsymbol{x}) = |u_0|_{L_\infty}$ in this domain. □

Remark. If the initial state u_0 is a odd 2ℓ-periodic C^1-smooth function; then there exist finite $C(u_0) = \sup_{\boldsymbol{x}\in[0,\ell]^n}\frac{|u_0(\boldsymbol{x})|}{\prod_{j=1}^n\sin\frac{\pi x_j}{\ell}}$ and thereby, repeating the proof of the previous Lemma, we obtain:

$$|u(t,\boldsymbol{x})| \leqslant C(u_0)\, e^{-nt\nu\frac{\pi^2}{\ell^2}}\prod_{j=1}^n\sin\tfrac{\pi x_j}{\ell}\,.$$

Later on we will show that even if a 2ℓ-periodic initial state u_0 is just in the L_∞-space, then the corresponding solution becomes C^∞-smooth for any $t > 0$, hence we can apply argument above, taking $u(t_0,\cdot)$ with some $t_0 > 0$ (e.g. $t_0 = 1$) as the initial state. The result is

$$|u(t,\boldsymbol{x})| \leqslant \tilde{C}(u_0)\, e^{-nt\nu\frac{\pi^2}{\ell^2}}\,.$$

Lemma. *For any 2ℓ-periodic solution u of equation (4.4.2), we have*

$$\tfrac{1}{2}\tfrac{d}{dt}\|u\|_k^2 \leqslant \|u\|_k^2\left(-\nu\big(\tfrac{\|u\|_k}{\|u\|_0}\big)^{2/k} + C_{k,p,n}|u_0|_{L_\infty}^{2p}\right), \tag{4.4.8}$$

where $C_{k,p,n} = 4^{k^2}(2p+1)^k n^k$.

Proof. The interpolation inequality $\|u\|_k \leqslant \|u\|_0^{\frac{1}{k+1}} \|u\|_{k+1}^{\frac{k}{k+1}}$ implies the inequality $\|u\|_{k+1} \geqslant \|u\|_k \big(\frac{\|u\|_k}{\|u\|_0}\big)^{\frac{1}{k}}$. Now (4.4.8) follows from (4.2.23) and (4.4.5). □

Corollary. Any odd 2ℓ-periodic solution of equation (4.4.2) satisfies the inequality:

$$\tfrac{1}{2}\tfrac{d}{dt}\|u\|_k^2 \leqslant \|u\|_k^2 \left(-\nu \Big(\tfrac{\|u\|_k e^{\nu \frac{\pi^2}{\ell^2} nt}}{\|u_0\|_0} \Big)^{2/k} + C_{k,p,n} |u_0|_{L_\infty}^{2p} \right) . \tag{4.4.9}$$

Proof. This inequality follows from (4.4.8) and (4.4.4). □

Theorem 4.4.1. *Let u be a 2ℓ-periodic solution of (4.4.2). Then for any positive ν and for any $t \geqslant 0$, we have*

$$\|u(t,\cdot)\|_k \leqslant \max\left\{ \|u_0\|_k \, , \, |u_0|_{L_2} \left(\tfrac{C_{k,p,n}|u_0|_{L_\infty}^{2p}}{\nu} \right)^{k/2} \right\}. \tag{4.4.10}$$

Let u be an odd 2ℓ-periodic solution of (4.4.2). Then for any positive ν and for any $t \geqslant 0$ we have

$$\|u(t,\cdot)\|_k \leqslant \max\left\{ \|u_0\|_k \, , \, |u_0|_{L_2} \left(\tfrac{C_{k,p,n}|u_0|_{L_\infty}^{2p}}{\nu} + \tfrac{\pi^2}{\ell^2} n \right)^{k/2} \right\} e^{-\nu \frac{\pi^2}{\ell^2} nt}. \tag{4.4.11}$$

Here $C_{k,p,n} = 4^{k^2}(2p+1)^k n^k$.

Proof. We start the proof with the odd 2ℓ periodic case. We denote

$$f(t) = \|u(t,\cdot)\|_k^2 \, e^{2\nu \frac{\pi^2}{\ell^2} nt} ; \quad B = |u_0|_{L_2}^2 \left(\tfrac{C_{k,p,n}|u_0|_{L_\infty}^{2p}}{\nu} + \tfrac{\pi^2}{\ell^2} n \right)^k .$$

Then (4.4.11) is equivalent to

$$f(t) \leqslant \max\{ f(0) \, , \, B \}.$$

First we show that if $f(t) > B$ then $f(t)$ is decreasing:

$$\tfrac{1}{2}\tfrac{d}{dt} f(t) = e^{2\nu \frac{\pi^2}{\ell^2} nt} \Big(\tfrac{1}{2}\tfrac{d}{dt} \|u(t,\cdot)\|_k^2 + \nu \tfrac{\pi^2}{\ell^2} n \, \|u(t,\cdot)\|_k^2 \Big) \overset{(4.4.9)}{\leqslant}$$
$$e^{2\nu \frac{\pi^2}{\ell^2} nt} \, \|u(t,\cdot)\|_k^2 \left(-\nu \Big(\tfrac{f(t)^{1/k}}{\|u_0\|_0^{2/k}} \Big) + C_{k,p,n} \, |u_0|_{L_\infty}^{2p} + \nu \tfrac{\pi^2}{\ell^2} n \right) < 0.$$

From this it follows that $f(t)$ can never be greater than $\max\{f(0), B\}$.

The proof of (4.4.10) for the periodic case is identical to this. □

Remark. Similarly to (4.4.11) we can show that for any odd 2ℓ-periodic solution of (4.4.2) for any positive ν, $t \geqslant 0$ and $\varepsilon \in [0,1]$ we have

$$\|u(t,\cdot)\|_k \leqslant \max\left\{ \|u_0\|_k \, , \, |u_0|_{L_2} \left(\tfrac{C_{k,p,n}|u_0|_{L_\infty}^{2p}}{\nu} + \tfrac{\pi^2}{\ell^2} \varepsilon n \right)^{k/2} \right\} e^{-\nu \frac{\pi^2}{\ell^2} \varepsilon nt}. \tag{4.4.12}$$

We can make the term involving the L_2 norm of u_0 above smaller by reducing ε, but this will increase the exponential coefficient. This will reduce the r.h.s. of (4.4.11) for small enough t.

Theorem 4.4.2. *Let u be a 2ℓ-periodic solution of (4.4.2). Then for any positive ν, for any $t \geqslant 0$, we have:*

$$\|u(t,\cdot)\|_k \leqslant \tfrac{\|u_0\|_0}{\nu^{k/2}} \max\left\{ \big(\tfrac{k}{t}\big)^{k/2} , \, (2C_{k,p,n})^{\frac{k}{2}} |u_0|_{L_\infty}^{pk} \right\}, \tag{4.4.13}$$

where $C_{k,p,n} = 4^{k^2}(2p+1)^k n^k$.

Proof. We denote $y(t) = \|u(t,\cdot)\|_k^2$. Suppose that

$$y(t) > \|u_0\|_0^2 \, \big(\tfrac{2C_{k,p,n}}{\nu} \, |u_0|_{L_\infty}^{2p} \big)^k, \tag{4.4.14}$$

otherwise there is nothing to prove. Then from (4.4.8) it follows that y is decreasing. Hence, (4.4.14) holds on the whole interval $[0,t]$. Using (4.4.8) again, we get

$$\tfrac{1}{2}\tfrac{d}{dt} y \leqslant -\tfrac{1}{2} \tfrac{\nu}{\|u_0\|_0^{2/k}} y^{1+\frac{1}{k}} .$$

Solving this differential inequality, we obtain $y(t) \leqslant \frac{k^k \|u_0\|_0^2}{\nu^k t^k}$. □

Remark. This is *not* actually a proof of the smoothness property for equation (4.4.2): *Take any initial state $u_0 \in L_\infty$; then there is a unique solution of (4.4.2) in the L_∞ space, which is C^∞-smooth for $t > 0$.* Here we have just proved that if $u(t,x)$ is a C^{k+1}-smooth function for $t > 0$ and satisfies (4.4.2) then it satisfies (4.4.14); but the smoothness property is well known, see [11], [29].

Remark. In the odd periodic case we can combine the last two theorems to obtain an exponential decay of the k^{th} Sobolev norm when the initial state is just in the L_∞ space.

Appendix A

Some technical inequalities

A.1 Two estimates for the sharp constants in the interpolation inequality

The Landau problem concerning sharp inequalities between the supremum norms of derivatives

$$\left|u^{(l)}\right|_{L_\infty} \leqslant C_{k,l}\left|u\right|_{L_\infty}^{\frac{k-l}{k}}\left|u^{(k)}\right|_{L_\infty}^{\frac{l}{k}}, \qquad 0 \leqslant l \leqslant k \tag{A.1.1}$$

on $(-\infty,\infty)$ was first completely solved by Kolmogorov in late thirties (see [14]).

Here we prove the estimates

$$\left(C_{k,2}\right)^k \leqslant 3 \quad \text{and} \quad \left(C_{k,1}\right)^k \leqslant 2$$

for the best constants in inequality (A.1.1). These constants are:

$$C_{k,l} = \frac{K_{k-l}}{K_k^{1-\frac{l}{k}}}, \quad \text{where} \quad K_i = \frac{4}{\pi}\left(1+\frac{(-1)^{i+1}}{3^{i+1}}+\frac{1}{5^{i+1}}+\frac{(-1)^{i+1}}{7^{i+1}}+\cdots\right). \tag{A.1.2}$$

Lemma A.1.1. *For $k \geqslant 2$ we have the inequality*

$$\left(C_{k,2}\right)^k \leqslant 3\,. \tag{A.1.3}$$

Proof. For $l=2$ the k^{th} power of expression (A.1.2) becomes:

$$\left(C_{k,2}\right)^k = \frac{16}{\pi^2}\cdot\frac{\left(1+\frac{(-1)^{k-1}}{3^{k-1}}+\frac{1}{5^{k-1}}+\frac{(-1)^{k-1}}{7^{k-1}}+\cdots+\frac{(-1)^{j(k-1)}}{(2j+1)^{k-1}}+\cdots\right)^k}{\left(1+\frac{(-1)^{k+1}}{3^{k+1}}+\frac{1}{5^{k+1}}+\frac{(-1)^{k+1}}{7^{k+1}}+\cdots+\frac{(-1)^{j(k+1)}}{(2j+1)^{k+1}}+\cdots\right)^{k-2}}\,.$$

When k is even, the signs alternate. When k is odd, all terms are positive.

For even k we have: $\left(C_{k,2}\right)^k < \frac{16}{\pi^2} < 3$. Using the identities $\sum\limits_{s=1}^{\infty}\frac{1}{s^2} = \frac{\pi^2}{6}$ and $\sum\limits_{s=1}^{\infty}\frac{1}{s^4} = \frac{\pi^4}{90}$, we obtain

$$\left(C_{3,2}\right)^3 = 3\,.$$

Let k be odd and greater than or equal to 3. Then using the Cauchy-Schwarz inequality, we obtain

$$\frac{C_{k,2}^k}{C_{k+2,2}^{k+2}} = \left(\frac{\left(1+\frac{1}{3^{k-1}}+\frac{1}{5^{k-1}}+\cdots\right)\left(1+\frac{1}{3^{k+3}}+\frac{1}{5^{k+3}}+\cdots\right)}{\left(1+\frac{1}{3^{k+1}}+\frac{1}{5^{k+1}}+\cdots\right)^2}\right)^k > 1\,. \qquad \square$$

Lemma A.1.2. *For $k \geqslant 1$ we have the inequality*

$$\left(C_{k,1}\right)^k \leqslant 2\,. \tag{A.1.4}$$

Proof. For $k=1$ we have

$$\left(C_{1,1}\right)^1 = \frac{\frac{4}{\pi}(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots)}{1} = \frac{\frac{4}{\pi}\frac{\pi}{4}}{1} = 1\,.$$

For $k=2$ we have

$$\left(C_{2,1}\right)^2 = \frac{\frac{4}{\pi}(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots)^2}{1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\cdots} = \frac{\frac{4}{\pi}(\frac{\pi^2}{8})^2}{\frac{\pi^3}{32}} = 2\,.$$

Let k be an odd number, then

$$\left(C_{k,1}\right)^k = \frac{\frac{4}{\pi}(1-\frac{1}{3^k}+\frac{1}{5^k}-\frac{1}{7^k}+\cdots)^k}{(1+\frac{1}{3^{k+1}}+\frac{1}{5^{k+1}}+\frac{1}{7^{k+1}}+\cdots)^{k-1}} < \frac{4}{\pi} < 2\,.$$

Suppose k is an even number; then

$$\left(C_{k,1}\right)^k = \frac{\frac{4}{\pi}(1+\frac{1}{3^k}+\frac{1}{5^k}+\frac{1}{7^k}+\cdots)^k}{(1-\frac{1}{3^{k+1}}+\frac{1}{5^{k+1}}-\frac{1}{7^{k+1}}+\cdots)^{k-1}}\,.$$

Fact. *The function $x \mapsto \left(1+\frac{1}{x}\right)^x$ is an increasing function for positive x; the function $x \mapsto \left(1+\frac{1}{x}\right)^{x+1}$ is an decreasing function for positive x.*

Proof. We calculate the logarithmic derivatives of these functions

in the first case we have $\frac{d}{dx}\big(x\log(1+\frac{1}{x})\big) = \log(x+1)-\log(x)-\frac{1}{x+1}$;

in the second case we have $\frac{d}{dx}\big((x+1)\log(1+\frac{1}{x})\big) = \log(x+1)-\log(x)-\frac{1}{x}$.

The proof concludes with the remark that for a concave smooth function f (*i.e.*, $f'' \leqslant 0$) we have

$$f(x+1) - f(x) - f'(x+1) \geqslant 0 \quad \text{and} \quad f(x+1) - f(x) - f'(x) \leqslant 0\,.$$

We apply this remark with $f(x) = \log(x)$. □

Next we use the simple fact that if a function f is a convex function ($f'' \geqslant 0$), then

$$f(j) \leqslant \frac{1}{2}\int_{j-1}^{j+1} f(x)dx.$$

From this we have

$$\frac{1}{5^k} + \frac{1}{7^k} + \frac{1}{9^k} + \cdots \leqslant \frac{1}{2}\int_4^\infty \frac{1}{x^k}dx = \frac{1}{2}\frac{1}{k-1}\frac{1}{4^{k-1}}.$$

Now for even $k \geqslant 4$ we have

$$(C_{k,1})^k \leqslant \frac{\frac{4}{\pi}\left(1+\frac{2}{3^m}\right)^m}{\left(1-\frac{1}{3^{m+1}}\right)^{m-1}} = \frac{4}{\pi}\left(1+\frac{2}{3^m}\right)^m\left(1+\frac{1}{3^{m+1}-1}\right)^{m-1} \leqslant \frac{4}{\pi}\left(1+\frac{2}{3^m}\right)^{2m} =$$

$$\frac{4}{\pi}\left(1+\frac{2}{3^m}\right)^{\frac{3^m}{2}\frac{4m}{3^m}} \leqslant \frac{4}{\pi}e^{\frac{4m}{3^m}} \leqslant \frac{4}{\pi}e^{\frac{1}{4}} < 2.$$

This completes the proof o the lemma. □

A.2 An interpolation inequality for the complex valued functions

Lemma A.2.1. *Let u be a C^2-smooth complex valued function defined on the interval $[0,\ell]$. Let M_0 and M_2 be positive real numbers such that $[u]_0 \leqslant M_0$ and $[u]_2 \leqslant M_2$. Then we have*

$$[u]_1 \leqslant \tfrac{2M_0}{\ell} + \tfrac{\ell M_2}{2} \qquad \text{if} \quad \ell^2 \leqslant \tfrac{4M_0}{M_2}\,, \tag{A.2.1}$$

$$[u]_1 \leqslant 2M_0^{1/2}M_2^{1/2} \qquad \text{if} \quad \ell^2 \geqslant \tfrac{4M_0}{M_2}\,. \tag{A.2.2}$$

Proof. Let $\mathfrak{U}_c(\ell, a_0, a_2)$ be the set of all C^2-smooth complex-valued functions u on the interval $[0,\ell]$ such that $[u]_0 \leqslant a_0$ and $[u]_2 \leqslant a_2$ on this interval. We denote

$$\Psi_c(\ell, a_0, a_2) = \sup\big\{[u]_1 : u \in \mathfrak{U}_c(\ell, a_0, a_2)\big\}.$$

Similarly, for real-valued functions on $[0,\ell]$ we define the function $\Psi_r(\ell, a_0, a_2)$. We note that both Ψ_r and Ψ_c are monotonic (non-decreasing) functions in a_0 and a_2. First we show that

$$\Psi_r(\ell, a_0, a_2) = \Psi_c(\ell, a_0, a_2). \tag{A.2.3}$$

Indeed, it is obvious that $\Psi_r(\ell, a_0, a_2) \leqslant \Psi_c(\ell, a_0, a_2)$, since $\mathfrak{U}_r(\ell, a_0, a_2) \subset \mathfrak{U}_c(\ell, a_0, a_2)$. To prove the converse inequality we show that for any $u \in \mathfrak{U}_c(\ell, a_0, a_2)$ we have $[u]_1 \leqslant \Psi_r(\ell, a_0, a_2)$. Given any $u \in \mathfrak{U}_c(\ell, a_0, a_2)$ we find $x_0 \in [0,\ell]$ such that $\left|\frac{\partial u}{\partial x}(x_0)\right| = [u]_1$. Let $\theta = \arg\{\frac{\partial u}{\partial x}(x_0)\}$. Using the fact that the operations of taking the derivative, taking the real part of the function, and multiplying the function by a constant are commutative, we obtain: $\frac{\partial\, \Re\{e^{-i\theta}u\}}{\partial x}(x_0) = \left|\frac{\partial u}{\partial x}(x_0)\right|$. We denote $v = \Re\{ue^{-i\theta}\}$. Since $v \in \mathfrak{U}_r(\ell, a_0, a_2)$, we have

$$[u]_1 = [v]_1 \leqslant \Psi_r(\ell, a_0, a_2),$$

so (A.2.3) is proven.

It follows that it is sufficient to give the proof of the Lemma for the real case. Let $u \in \mathfrak{U}_r(\ell, a_0, a_2)$ and x_0 be the point where the maximum value of the first derivative of the function u is achieved. Without loss of generality we assume that $a_1 \stackrel{\text{def}}{=} \frac{\partial u}{\partial x}(x_0) > 0$. Consider the function

$$\psi_{x_0}(x) = u(x_0) + a_1(x - x_0) - \tfrac{a_2\,\mathrm{sign}(x-x_0)(x-x_0)^2}{2}.$$

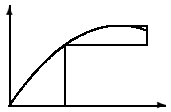
It follows that

$$\begin{aligned} &\text{for } x \in [x_0,\ell] \text{ we have} \qquad u(x) \geqslant \psi_{x_0}(x), \\ &\text{for } x \in [0,x_0] \text{ we have} \qquad u(x) \leqslant \psi_{x_0}(x). \end{aligned} \tag{A.2.4}$$

For positive real numbers l and a_2, and for arbitrary real number a_1 we define

$$h(l, a_1, a_2) = \sup_{x\in[0,l]}\big\{a_1x - \tfrac{a_2x^2}{2}\big\}.$$

It is clear that this function is non-negative and non-decreasing with a_1 and l. This function is equal to zero for $a_1 \leqslant 0$ and increasing with a_1 for $a_1 > 0$. Also we have



$$h(l_1 + l_2, a_1, a_2) = h(l_1, a_1, a_2) + h(l_2, a_1 - l_1 a_2, a_2).$$

Hence

$$h(l_1 + l_2, a_1, a_2) \leqslant h(l_1, a_1, a_2) + h(l_2, a_1, a_2). \tag{A.2.5}$$

It follows that

$$\begin{aligned} 2a_0 \geqslant \max_{x\in[0,\ell]} u - \min_{x\in[0,\ell]} u \geqslant \max_{x\in[x_0,\ell]} u - \min_{x\in[0,x_0]} u \overset{(A.2.4)}{\geqslant} \max_{x\in[x_0,\ell]} \psi_{x_0} - \min_{x\in[0,x_0]} \psi_{x_0} \\ \geqslant h(\ell - x_0, a_1, a_2) + h(x_0, a_1, a_2) \overset{(A.2.5)}{\geqslant} h(\ell, a_1, a_2). \end{aligned} \tag{A.2.6}$$

Let $a(\ell, h, a_2)$ be the inverse of the function $h(\ell, a_1, a_2)$ with respect to a_1, for fixed ℓ and a_2. The function h is invertible for $a_1 > 0$. (Since it is increasing there). The value of the inverse function $a(\ell, h, a_2)$ is actually the maximum value of the derivative $u'(0)$ over the family of the quadratic functions $u : [0, \ell] \to \mathbb{R}$, $\quad \{x \mapsto ax - \frac{a_2 x^2}{2}\}$ with the restriction $\sup\limits_{[0,\ell]} u(x) \leqslant h$. We see that

$$a(\ell, h, a_2) = \begin{cases} \sqrt{2ha_2} & \text{if } \ell \geqslant \sqrt{\frac{2h}{a_2}}, \\ \frac{h}{\ell} + \frac{a_2 \ell}{2} & \text{if } \ell \leqslant \sqrt{\frac{2h}{a_2}}. \end{cases}$$

Indeed, consider the first, i.e., the smallest value of the parameter a, such that the parabola $ax - \frac{a_2 x^2}{2}$ meets the segment $\big[(0, h); (\ell, h)\big]$. The intersection point is either an interior point or the end-point (ℓ, h). The first case occurs if $\ell > \sqrt{2h/a_2}$. In this case we have $a = \sqrt{2ha_2}$. The second case occurs if $\ell \leqslant \sqrt{2h/a_2}$ and we have $a = h/\ell + a_2\ell/2$.

Using the fact that the function $a(\ell, h, a_2)$ is monotonic in h we have from (A.2.6) that

$$a(\ell, 2a_0, a_2) \geqslant a\big(\ell, h(\ell, a_1, a_2), a_2\big) = a_1 .$$

We arrive at (A.2.1) and (A.2.2). □

Corollary. *Under the conditions of the previous Lemma, the inequality*

$$[u]_1 \leqslant \frac{2\,[u]_0}{\ell} + 2\,[u]_0^{1/2}\,[u]_2^{1/2} \ . \tag{A.2.7}$$

holds for any $\ell > 0$.

Proof. If $\ell^2 \geqslant \frac{4[u]_0}{[u]_2}$ then this follows from (A.2.2). If $\ell^2 \leqslant \frac{4[u]_0}{[u]_2}$ then (A.2.7) follows from (A.2.1) and the inequality $\frac{\ell[u]_2}{2} \leqslant [u]_0^{1/2}\,[u]_2^{1/2}$. □

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