Comments on the paper

On Invariant Measures of the 2D Euler Equation

by A. Biryuk

The paper under consideration is devoted to the study of certain invariant measures for the 2D Euler equations on the torus. The author argues that the invariant measures obtained by Kuksin in [K04] form a natural family of invariant measures for the Euler equations since they arise as limits of invariant measures for the 2D Navier-Stokes equations under stochastic forcing. Another natural family of measures can be obtained by taking convex linear combinations of 'microcanonical' measures on the surfaces of constant energy and enstrophy for an N-dimensional Galerkin approximation to the true Euler equation and then let the cut-off N in the approximation go to infinity. The latter construction is often performed in the physics literature. The point made by the author is that these two classes of measures disagree since the $\mathcal{H}^2$ Sobolev-norm of the velocity field u is bounded in the first case (under some assumptions on the stochastic forcing) and diverges as $N \to \infty$ in the second case. This divergence is of course well-known and prompts statements of the type [KM80]: "Like all classical fields a velocity field obeying the inviscid Navier-Stokes (Euler) equation exhibits an ultraviolet catastrophe. Normalisable inviscid equilibrium states are obtained only if the system is truncated by eliminating high-wavenumber modes."

The author states that the physics literature often confuses the two families of invariant measures constructed above and formulates this as Conjecture 5, which he then sets out to disprove along the lines sketched above. The fact that this statement can be disproved may be of some interest to the physics community, although they seem to be aware of it, since the quotation above is followed by the statement that "The equilibrium statistics which we shall now discuss do not have a direct pertinence to typical turbulence, which is far from absolute equilibrium".

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A part of the review is skipped, since it is no longer relevant to the final version.

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Note that I do believe that Theorem 6 holds, because it is easy to explain it at a heuristic level. It is well-known from the work of Talagrand and others that in a certain sense, the centred Gaussian measure on $\mathbb{R}^N$ with covariance $1/N$ times the identity comes 'close' to the Lebesgue measure on the unit sphere in $\mathbb{R}^N$. It follows that the measure $\ell^N_{c_1,c_2}$ should be 'close' (in a sense to be made precise) to the measure with density $C\exp\big(-n(N)(\alpha E+\beta\Omega)\big)$ with respect to Lebesgue measure, for some constants α, β. The asymptotic behaviour as $N \to \infty$ of equation (8) of the article under consideration follows immediately.

Reviewed by an anonymous referee.