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Lower bounds for derivatives of solutions for NLS

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Abstract. The main goal of this paper is to obtain the following lower bounds

$$|D_x^m u_\nu(\cdot,\cdot)|_{L^\infty\left([0,\nu^{-1/3}]\times\Omega\right)} \geqslant C_m \nu^{-m/3}$$

for solutions u_ν of nonlinear Schrödinger (NLS) equations with small viscosity ν. We also discuss the application of the above estimate to the theory of turbulence. Namely, we are interested in time-averaged lower bounds, which are important in establishing upper bounds for the turbulent space scale.

1 Introduction

In the late nineties (late 1990s) S. Kuksin established a range of results related to the theory of turbulence (NLS-turbulence), see [8, 9, 10] and references therein. In particular, he proved that for an arbitrary positive ε we have the following lower bounds

$$|D_x^m u_\nu(\cdot,\cdot)|_{L^\infty\left([0,\nu^{-1/3}]\times\Omega\right)} \geqslant \nu^{-\left(\frac{1}{3}-\varepsilon\right)m} \tag{1.1}$$

for solutions u_ν of nonlinear Schrödinger (NLS) equations with small viscosity $\nu < \nu_{\varepsilon,m}$, $m \geqslant 2$, see [10, theorem 3]. Some of disadvantages of the result in [10] are that the above inequality holds non-uniformly in ε and m and the dependence of $\nu_{\varepsilon,m}$ on ε and m is not specified. The main goal of this paper is to establish similar bounds without the ε appearing above. In addition, our version of the inequality holds uniformly in m for $m \geqslant 2$. A similar result for generalized multidimensional Burgers equations was obtained in [2]. We also revisit the application of these bounds to turbulence. Namely, we rederive the time-averaged lower bounds for Sobolev seminorms, which are important in establishing upper bounds for the turbulent space scale.

Let Ω be an open connected domain in $\mathbb{R}^n$ or a periodic domain (torus). Let z be a fixed complex unitary number, i.e., $|z| = 1$. Let p be a fixed natural number. Let $u = u(t, \boldsymbol{x})$ be a complex valued function that satisfies equation

$$-i\dot{u} = -\nu z \Delta u \pm |u|^{2p}\, u\,. \tag{1.2}$$

in the interior of $[0,\infty)\times\Omega$. Here $\nu > 0$ is a small positive real number (viscosity, or the turbulence index). The "±" sign indicates that we are considering both the focusing and the defocusing cases. In fact the technique used and the results obtained are the same for both cases.

Our main theorem is the following.

Theorem 1.1. *Suppose that u is a smooth solution of (1.2) with $p \geqslant 1$ on $[0,\infty)\times\Omega$, with a smooth initial state u_0. No boundary conditions for u on $\partial\Omega$ are assumed. Suppose, that there is a real number $c > 0$ such that*

$$\operatorname{osc}_\Omega |u_0(x)| \overset{\text{def}}{=} \sup_{x,y\in\Omega} \big(|u_0(x)| - |u_0(y)|\big) \geqslant c\,. \tag{1.3}$$

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Then for every constant $c_t > 0$ there exists a positive constant $\nu' = \nu'(|u_0|_{L_\infty}, c, c_t)$, and ν-independent positive constants $C_m = C_m(|u_0|_{L_\infty}, c, c_t)$, $m = 2, 3, \dots$ and a time $t^\star = t^\star(u_0, \nu, c_t)$, $0 \leqslant t^\star \leqslant c_t \nu^{-\frac{1}{3}}$ such that for $m \geqslant 2$

$$[u(t^\star)]_{C^m(\Omega)} \geqslant C_m \nu^{-m/3} \qquad \text{for all positive } \nu < \nu' \tag{1.4}$$

and

$$\operatorname{osc}_\Omega |u(t, \cdot)| \geqslant \tfrac{c}{2} \qquad \text{and} \qquad |u(t, \cdot)|_{L_\infty} \leqslant |u_0|_{L_\infty} + \tfrac{c}{2} \qquad \text{for all } t \in [0, t^\star].$$

Moreover, the dependence of C_m and ν' on $|u_0|_{L_\infty}$, c and c_t can be chosen to be continuous.

Here $[u]_{C^m} = \sup \max\limits_{|\alpha|=m} \left| \frac{\partial^\alpha}{\partial x^\alpha} u \right|$ denotes the classical C^m semi-norm. The constants C_m and ν' also depend on the spacial dimension n and the domain Ω, but we suppress this in the notation to avoid cumbersomeness. Explicit values for the constants $C_m = C_m(\min|u_0|, \max|u_0|, c_t)$ and ν' can be easily extracted from the proof. Among other advantages is that we have no restriction on the space dimension. Also, we have no restriction neither on the real nor on the imaginary part of z. The constants C_m in theorem 1.1 are generally less than one. In inequality (1.1) the initial constant is absorbed by imperfection of the power of the viscosity parameter ν. We can obtain inequality (1.1) from theorem 1.1 (with $c_t = 1$) by choosing $\nu_{\varepsilon,m} = \min\{(C_m)^{1/\varepsilon m}, \nu'\}$.

The theorem can be applied as follows. For any $C > c$ we can deduce uniform lower bounds similar to (1.4) (but with reduced C_m and ν') for any family of solutions, corresponding to a family of initial conditions $\{u_0 : |u_0|_{L_\infty} \leqslant C, \operatorname{osc}|u_0| \geqslant c\}$. To achieve the uniform bounds it is sufficient to take the minimum of C_m and ν' over $c \leqslant |u_0|_{L_\infty} \leqslant C$. This is possible since the dependence of constants C_m and ν' on $|u_0|_{L_\infty}$ is continuous. This proves that solutions of NLS with large Reynolds number have short space-scale, see [10]. Another application of the theorem is in the development of the theory of solutions in low regularity spaces [3, 4]. The following idea behind the proof is due to Kuksin [10]. The fact that the zero dispersive limit ($\nu = 0$) can be solved explicitly, enables us to analyze the original equation with a Laplacian term using perturbation theory[2]. The same idea was used later in [3, 4] to prove local ill-posedness for the Shrödinger equation in low-order Sobolev spaces. In [2] this idea was applied to multidimensional Burgers equations and their generalizations. Lower bounds for the m-th derivative of the form $\nu^{-m/2}$ were obtained. It is not clear whether we can employ this idea for the Navier-Stokes system. The reason is, in part, that the zero viscosity limit, the Euler equation, is even more difficult to study. However, the Navier-Stokes system can be viewed as a perturbed Burgers equation. No growth rate has been obtained yet in this way, though this approach implies that derivatives of solutions to the Navier-Stokes system are separated away from zero, uniformly in small viscosity ν, if the initial state satisfies a certain nondegeneracy condition, see [1].

Our second theorem improves previously known lower bounds for time averaged Sobolev seminorms. This improvement becomes possible due to theorem 1.1 and also due to a refined approach to obtain lower estimates for time-averaged Sobolev seminorms when it is known that the classical derivative is large at a single time-point. We use $\|\cdot\|_m$ to denote the m^{th} Sobolev seminorm, see (2.1).

Theorem 1.2. *Assume that Ω is a torus and that $\operatorname{Im} z \geqslant 0$. Under the assumptions of theorem 1.1 there exist ν-independent constants C'_m such that for all sufficiently large m and sufficiently small ν we have*

$$\nu^{1/3} \int_0^{\nu^{-1/3}} \|u\|_m^2 \, \mathrm{d}t \geqslant C'_m \nu^{-\frac{2}{3}\left(m - \frac{n+1}{2}\right)}. \tag{1.5}$$

This theorem will be proved in section 4. We employ several ideas from [10], simplifying some arguments. Among other advantages of theorem 1.2 is that solutions u are not assumed to be odd. It is worth noting that the conclusion of theorem 1.2 does not follow from the conclusion of theorem 1.1 alone. Indeed, as a counterexample one could consider a function highly peaked in time. Some extra arguments and information about the function u should be used. In our case we again use the fact that the function u satisfies equation (1.2) which enables us to use Gagliardo-Nirenberg type inequalities to obtain some estimates for the Sobolev norms of u. For this approach to work we should assume that $\operatorname{Im} z \geqslant 0$.

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[2] However, a better implementation of this idea allows us to obtain a better result. The fact that the Laplacian term involves only the second derivatives is the reason why the theorem is applicable only for $m \geqslant 2$.

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2 Previous results revisited

In paper [10] upper and low estimates for the derivatives of solutions of (1.2) were studied. In particular the following result was obtained:

Theorem 2.1. (Theorem 3 in [10].) *Suppose that u is a smooth solution of (1.2) with $p > 1$ on $[0, \infty) \times \Omega$, with a smooth initial state u_0. No boundary conditions for u on $\partial\Omega$ are assumed. Suppose, that*

$$\operatorname{osc}_\Omega |u_0(x)| \overset{\text{def}}{=} \sup_{x,y\in\Omega} \big(|u_0(x)| - |u_0(y)|\big) \geqslant 1\,.$$

Then for any $\kappa < \frac{1}{3}$ and $m \geqslant 2$ there exists a positive constant $\nu' = \nu'(\kappa, m, |u_0|_{L_\infty})$ and a time $t^\star = t^\star(u_0)$, $0 \leqslant t^\star \leqslant \nu^{-\frac{1}{3}}$ such that

$$|u(t^\star)|_{C^m(\Omega)} \geqslant \nu^{-m\kappa} \qquad \text{for all positive } \nu < \nu'$$

and $\operatorname{osc}_\Omega |u(t^\star, \cdot)| \geqslant \frac{1}{2}$.

The time interval of the form $0 \leqslant t^\star \leqslant \nu^{-\frac{1}{3}}$ is chosen for convenience only. The same result remains true for $0 \leqslant t^\star \leqslant \mathrm{Const}\, \nu^{-\frac{1}{3}}$, if ν' is modified accordingly. The same applies for the condition $\operatorname{osc}_\Omega |u_0(x)| \geqslant 1$ (it can be changed by rescaling).

While our main goal is to show that the lower bound $\nu^{-m\kappa}$ above can be replaced with $C_m\nu^{-\frac{1}{3}m}$, i.e. $\kappa < \frac{1}{3}$ can be replaced with $\frac{1}{3}$, we will demonstrate a corollary to the previous theorem which is also due to Kuksin. It deals with the periodic solutions of (1.2), for which lower bounds for time average of the H^m-Sobolev seminorms $\|\cdot\|_m$ are obtained. The H^m-Sobolev seminorms $\|u\|_m$ for a periodic function u are defined as follows:

$$\|u\|_k^2 = \int_{\mathbb{T}^n} u\,(-\Delta)^k \bar{u}\ \mathrm{d}^n\boldsymbol{x}\,. \tag{2.1}$$

Here $\mathbb{T}^n$ denotes the periodicity domain of the function u (a torus). We note that $\|u\|_0 = |u|_{L^2}$.

In the statement of this theorem, we have made explicit the exponent of ν in (2.2) implied by the proof. This exponent is not explicitly stated in [10]. This enables us to quantify the improvements on the bounds arising from the different approach taken in the current paper.

Theorem 2.2. (see [10, Thm 4]) *Let $\Omega = \mathbb{T}^n$ be a periodic domain (torus). Let $p \in \mathbb{N}$. Assume that $\operatorname{Im} z \geqslant 0$. Let the smooth function $u = u_\nu(t, x)$ satisfy (1.2) for $t > 0$, $\|u_\nu(0,\cdot)\|_m \leqslant C_m$ for $m \geqslant 0$, and $\operatorname{osc}|u_\nu(0,\cdot)| \geqslant 1$. Then for any integer $m \geqslant \max\{\left[\frac{n}{2}\right] + 3, pn\}$ and for any $\varepsilon > 0$ there exists $\nu_{\varepsilon,m}$ (which also depends on C_m) such that for any positive $\nu < \nu_{\varepsilon,m}$ we have*

$$\nu^{1/3}\int_0^{\nu^{-1/3}} \|u\|_m^2\ \mathrm{d}t \geqslant \nu^{-\frac{2}{3}(m-\left[\frac{n}{2}\right]-2-\varepsilon)\frac{m}{m+pn}}\,. \tag{2.2}$$

Here $\left[\frac{n}{2}\right]$ denotes the integer part of $\frac{n}{2}$. In particular, $\left[\frac{n}{2}\right] = 1$ for $n = 2$ and for $n = 3$.

Remark 2.3. *Theorem 4 in [10] is proven in dimensions $n \leqslant 3$ only[3] and for $m \geqslant \max\{\left[\frac{n}{2}\right] + 3, pn+1\}$. However, the condition $m \geqslant pn+1$ was lost in both the statement and the proof. Careful track of exponents in [10] reveals the following lower bound: $\nu^{-\frac{2}{3}(m-\left[\frac{n}{2}\right]-2-\varepsilon)\frac{m}{m+pn+1}}$. Our form of the key ingredient* (2.4) *makes the proof shorter and more transparent. Moreover, new lemma 2.4 allows us to get a slightly better exponent and a better condition on m. Furthermore, we are not restricted to the case of odd functions.*

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[3] Because [10] is devoted to the case $n \leqslant 3$. However the paper can be easily extended to the case of any spacial dimension n.

Proof of theorem 2.2 given theorem 2.1. Let $z = a + bi$, $b \geqslant 0$. The function $u_\nu(t, x)$ solves the equation

$$\partial_t u = (b - ai)\nu\Delta u \pm i|u|^{2p}u. \tag{2.3}$$

Multiplying (2.3) by $\bar{u}$, integrating over the period, and taking the real part we obtain $\frac{1}{2}\partial_t\|u\|_0^2 = -b\nu\|u\|_1^2 \leqslant 0$. Hence the L^2-norm $\|u\|_0$ is non-increasing with time and therefore is bounded by C_0. Similarly, we multiply (2.3) by u in H^m, i.e., we multiply (2.3) by $(-\Delta)^{2m}\bar{u}$, integrate over the periodicity domain, and take the real part. The result is

$$\tfrac{1}{2}\partial_t\|u\|_m^2 = -b\nu\|u\|_{m+1}^2 - \operatorname{Im}\left\langle |u|^{2p}u, u\right\rangle_m \leqslant \left\||u|^{2p}\,u\right\|_m \|u\|_m .$$

Hence $\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_m \leqslant \left\||u|^{2p}\,u\right\|_m$ and $\left(\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_m\right)_+ \leqslant \left\||u|^{2p}\,u\right\|_m$, where $(a)_+ = \max\{0, a\}$.
One of the key ingredients in the proof is the following fact:
Let $f \in C^1([0, T]; \mathbb{R})$ (*or let* f *be Lipschitz or even only an absolutely continuous function*) *then*

$$\int_0^T \left(\tfrac{\mathrm{d}f}{\mathrm{d}t}(t)\right)_+ \,\mathrm{d}t \geqslant \sup_{t\in[0,T]} f(t) - f(0). \tag{2.4}$$

From the Sobolev inequality

$$\|u\|_{H^{\frac{n}{2}+\delta}} \geqslant c|u|_{L^\infty} \tag{2.5}$$

we have

$$\|u(t^\star, \cdot)\|_m \geqslant c|u(t^\star, \cdot)|_{C^{m-[\frac{n}{2}]-1}} \overset{\text{by Thm 2.1}}{\geqslant} c\nu^{-\frac{1}{3}(m-[\frac{n}{2}]-1-\varepsilon)} \tag{2.6}$$

for $m \geqslant \left[\frac{n}{2}\right] + 3$. Applying the "key ingredient" we obtain

$$\int_0^T \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\|u\|_m\right)_+ \,\mathrm{d}t \geqslant c\nu^{-\frac{1}{3}(m-[\frac{n}{2}]-1-\varepsilon)} \qquad \text{for small } \nu$$

since $\|u_\nu(0, \cdot)\|_m$ is bounded. Here $T = \nu^{-1/3}$ and "for small ν" means $\nu \in (0, \nu_{m,\varepsilon})$. To complete this proof we need the following lemma:

Lemma 2.4. *Let* $m > \frac{n}{2}$. *Then* $\left\||u|^{2p}\,u\right\|_m \leqslant C\left(\|u\|_0^{2p-\frac{pn}{m}}\,\|u\|_m^{1+\frac{pn}{m}} + \left|\langle u\rangle\right|^{2p}\|u\|_m\right)$, *where* $\langle u\rangle = \frac{1}{\operatorname{Vol}(\mathbb{T}^n)}\int_{\mathbb{T}^n} u\,\mathrm{d}\boldsymbol{x}$ *is the mean value of the function* u.

Proof. We first assume that $\langle u\rangle = 0$. Fix positive $\delta \leqslant \min\{\frac{n}{2}, m - \frac{n}{2}\}$. The proof then follows from the chain of inequalities:

$$\tfrac{1}{C_m}\left\||u|^{2p}\,u\right\|_m \leqslant \|u\|_m\,|u|_{L^\infty}^{2p} \leqslant C'_\delta\,\|u\|_m\,\|u\|_{\frac{n}{2}+\delta}^{p}\,\|u\|_{\frac{n}{2}-\delta}^{p} \leqslant C'_\delta\,\|u\|_m\,\|u\|_0^{2p-\frac{pn}{m}}\,\|u\|_m^{\frac{pn}{m}} .$$

The first inequality follows by multiple application of lemma A1 in appendix. The second inequality is the Sobolev interpolation for L^∞ (see appendix, lemma A2). The latter is just an interpolation of H^s norms (see appendix, lemma A3). The case $\langle u\rangle \neq 0$ follows by applying the full version of lemma A2 for a function with non-zero mean value and the inequality $(a + b)^k \leqslant 2^{k-1}(a^k + b^k)$ for $k = 2p \geqslant 1$. Lemma 2.4 is proven. $\square$

We now continue with the proof of theorem 2.2. Since the L^2-norm $\|u\|_0 \leqslant C_0$ is bounded for any $t \geqslant 0$ and $\left|\langle u\rangle\right| \leqslant \frac{1}{\sqrt{\operatorname{Vol}(\mathbb{T}^n)}}\|u\|_0$, we have from lemma 2.4:

$$\int_0^T \|u\|_m^{1+\frac{pn}{m}} + \|u\|_m \,\mathrm{d}t \geqslant c\nu^{-\frac{1}{3}(m-[\frac{n}{2}]-1-\varepsilon)}, \qquad \text{where } T = \nu^{-1/3}.$$

Hence, at least one of the following inequalities hold:

$$\nu^{1/3}\int_0^{\nu^{-1/3}} \|u\|_m^{1+\frac{pn}{m}} \,\mathrm{d}t \geqslant \tfrac{c}{2}\nu^{-\frac{1}{3}(m-[\frac{n}{2}]-2-\varepsilon)} \quad \text{or} \quad \nu^{1/3}\int_0^{\nu^{-1/3}} \|u\|_m \,\mathrm{d}t \geqslant \tfrac{c}{2}\nu^{-\frac{1}{3}(m-[\frac{n}{2}]-2-\varepsilon)}.$$

For $\alpha \leqslant 2$ we have $\frac{1}{T}\int_0^T f^2\,\mathrm{d}t \geqslant \left(\frac{1}{T}\int_0^T f^\alpha\,\mathrm{d}t\right)^{\frac{2}{\alpha}}$. Using this fact for $f(t) = \|u(t,\cdot)\|_m$ with $\alpha = 1 + \frac{pn}{m}$ and $\alpha = 1$ we finally get

$$\nu^{1/3}\int_0^{\nu^{-1/3}} \|u\|_m^2\,\mathrm{d}t \geqslant \tilde{c}\nu^{-\frac{2}{3}(m-[\frac{n}{2}]-2-\varepsilon)\frac{m}{m+pn}}.$$

Since $\varepsilon > 0$ is arbitrary, we can set the constant $\tilde{c}$ to be 1 by decreasing $\nu_{m,\varepsilon}$ if necessary. The condition $m \geqslant pn$ is used to ensure that $\alpha = 1 + \frac{pn}{m} \leqslant 2$. Theorem 2.2 is proven. $\square$

Using theorem 1.1 instead of theorem 2.1 as a basis for theorem 2.2 we can obtain an immediate improvement (see (2.10)). Indeed, the proof of theorem 2.2 is based on the following chain of inequalities

$$\frac{1}{T}\int_0^T \|u\|_m^2\,\mathrm{d}t \geqslant \left(\frac{1}{T}\int_0^T \|u\|_m^{1+\frac{pn}{m}}\,\mathrm{d}t\right)^{\frac{2}{\alpha}} \geqslant \left(\frac{1}{T}\sup \|u\|_m\right)^{\frac{2}{\alpha}} \tag{2.7}$$

where $\frac{2}{\alpha} = \frac{2m}{m+pn}$, and by using these in conjunction with theorem 1.1 we can avoid loss of a power of ν as we did using inequalities (2.5) and (2.6). Instead of (2.5) we use the following estimate which is sharp in terms of the powers

$$|v|_{L^\infty} \leqslant C_{n,\delta}\,\|v\|^{1/2}_{\frac{n}{2}-\delta}\,\|v\|^{1/2}_{\frac{n}{2}+\delta}\,. \tag{2.8}$$

Here we assume that the mean value of v is zero, see appendix, lemma A2. Applying this to derivatives $v = D^m u$ we obtain that the inequality

$$\|u(t^\star)\|_k \geqslant C_k \nu^{-(k-n/2)/3} \tag{2.9}$$

can not be violated for both $k = m + n/2 - \delta$ and $k = m + n/2 + \delta$ if $m \geqslant 2$. In fact we claim that (2.9) is true for all sufficiently large real k. Indeed, assuming the opposite, we find sufficiently large k_1 and k_2 such that $k_2 - k_1 > 2$ and

$$\|u(t^\star)\|_{k_j} < c_{k_j}\nu^{-(k_j-n/2)/3} \qquad j = 1,2,$$

where $c_{k_j} \ll 1$, $j = 1,2$ are to be chosen later. By interpolation we find that

$$\|u(t^\star)\|_k < c_k \nu^{-(k-n/2)/3} \qquad k \in [k_1,k_2].$$

Therefore using inequality

$$[u]_{C^m} \leqslant C\,\|u\|^{1/2}_{m+\frac{n}{2}-\delta}\,\|u\|^{1/2}_{m+\frac{n}{2}+\delta}$$

for a suitable m such that $m + \frac{n}{2} \mp \delta \in [k_1,k_2]$ we find $[u]_{C^m} <$ (small constant)$\nu^{-m/3}$. Choose the constants $c_{k_1} \ll 1$ and $c_{k_2} \ll 1$ so that the small constant above is less then the constant in (1.4). The contradiction obtained with theorem 1.1 proves that (2.9) holds for all sufficiently large k. Summarizing the arguments, i.e., plugging (2.9) into (2.7) with $T = \nu^{-1/3}$ we conclude that for all sufficiently large real m we have

$$\nu^{1/3}\int_0^{\nu^{-1/3}} \|u\|_m^2\,\mathrm{d}t \geqslant C_m \nu^{-\frac{2}{3}(m-\frac{n}{2}-1)\frac{m}{m+pn}}. \tag{2.10}$$

The statement of theorem 1.2 is even stronger. We postpone discussion of time-averaged Sobolev seminorms till section 4.

3 Proof of theorem 1.1

We start with a technical lemma, from which we then derive the main theorem.

Lemma 3.1. *Let $u = u(t,\boldsymbol{x})$ be a smooth complex valued function that satisfies equation (1.2) with $p > 1$ in the interior of the domain $[0,\infty)\times[0,\ell]^n$.*

Suppose that there exists a point $\boldsymbol{x}_1 \in [0,\ell]^n$ such that

$$0 < |u(0,\boldsymbol{x}_1)| \overset{\mathrm{def}}{=} a < \sup_{\boldsymbol{x}\in[0,\ell]^n} |u(0,\boldsymbol{x})| \overset{\mathrm{def}}{=} b\,. \tag{3.1}$$

Let $\sigma > 0$ be a real number such that the σ-neighborhoods of the numbers a and b do not intersect and do not contain zero, i.e., $\sigma < a$ and $\sigma < \frac{b-a}{2}$.

Suppose that the positive real numbers c_d and c_t satisfy the following inequalities:

$$n c_t c_d^2 \leqslant \sigma, \tag{3.2}$$

$$c_d < c_t \frac{\big((b-\sigma)^{2p} - (a+\sigma)^{2p}\big)(a-\sigma)}{4n\ell(b+\sigma)^{1/2}}, \tag{3.3}$$

and that ν is small enough to satisfy the following two inequalities:

$$c_d^2 \nu^{-2/3} \ell^2 \geqslant 4(b+\sigma)\,, \tag{3.4}$$

$$c_t \nu^{-1/3}\big((b-\sigma)^{2p} - (a+\sigma)^{2p}\big) \geqslant \frac{4n\ell(b+\sigma)^{1/2} c_d}{a-\sigma}\nu^{-1/3} + \frac{\sigma}{b-\sigma} + \frac{\sigma}{a-\sigma}\,. \tag{3.5}$$

Then there exists a time $t^\star \in [0,T]$, where $T = c_t \nu^{-1/3}$, such that

$$\max_{j=1,\ldots,n} \sup_{\boldsymbol{x}\in[0,\ell]^n} \left| \frac{\partial^2 u}{\partial x_j^2}(t^\star, \boldsymbol{x}) \right| \geqslant c_d^2 \nu^{-2/3}, \tag{3.6}$$

$$\sup_{\substack{\boldsymbol{x}\in[0,\ell]^n \\ t\in[0,t^\star]}} \big|u(t,\boldsymbol{x})\big| \leqslant b+\sigma\,, \tag{3.7}$$

$$\inf_{t\in[0,t^\star]} \operatorname{osc}|u(t,\cdot)| \geqslant b-a-2\sigma\,. \tag{3.8}$$

Remark 3.2. *We briefly discuss the purpose of conditions (3.2)–(3.5). Condition (3.2) is imposed in order to have the following property: The change in a Lipschitz function with the Lipschitz constant $nc_d^2\nu^{1/3}$ over a time interval of length $c_t\nu^{-1/3}$ is at most σ. The purpose of (3.3) is to make (3.5) possible. Condition (3.4) is in order to apply the multiplicative interpolation inequality (A.4). Inequality (3.5) is the main condition needed to prove the lemma.*

Remark 3.3. *From (3.2) and (3.3) it follows that c_d can be chosen to be continuous in a, b, σ and c_t. From (3.4) and (3.5) it follows that the smallness threshold for ν (the maximal admissible ν) depends continuously on all other parameters.*

Proof (of lemma 3.1). Let $\boldsymbol{x}_2$ be a point at which the supremum in (3.1) is achieved. For simplicity we assume that $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are internal points of the cube $[0,\ell]^n$. The general case is obtained by a limiting process. In what follows $[\boldsymbol{x}_1, \boldsymbol{x}_2]$ denotes the straight line segment joining $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. Without loss of generality, we assume that for all points $\boldsymbol{x} \in [\boldsymbol{x}_1, \boldsymbol{x}_2]$ we have $|u(0,\boldsymbol{x})| \in [a,b]$ $\big($otherwise we can consider a subsegment of $[\boldsymbol{x}_1, \boldsymbol{x}_2]\big)$.

We proceed by an argument by contradiction. Suppose that (3.6) does not hold, i.e.:

$$\left| \frac{\partial^2 u}{\partial x_j^2}(t, \boldsymbol{x}) \right| < c_d^2 \nu^{-2/3} \qquad \text{for all } j = 1,\ldots,n,\ t\in[0,T],\ \boldsymbol{x}\in[0,\ell]^n. \tag{3.9}$$

Then from the equation (1.2) it follows that for $t \in [0,T]$ the function $|u|(t,\boldsymbol{x})$ is Lipschitz in t with the Lipschitz constant $nc_d^2\nu^{1/3}$. Using this and inequality (3.2), we have

$$\big||u(0,\boldsymbol{x})| - |u(t,\boldsymbol{x})|\big| < \sigma \tag{3.10}$$

for all $\boldsymbol{x} \in [0,\ell]^n$ and $t \in [0,T]$. Note that using (3.1) inequality (3.10) implies $|u(t,\boldsymbol{x})| \leqslant b+\sigma$. Furthermore, (3.1) and (3.10) imply

$$a-\sigma \leqslant |u(t,\boldsymbol{x}_1)| \leqslant a+\sigma \qquad \text{and} \qquad b-\sigma \leqslant |u(t,\boldsymbol{x}_2)| \leqslant b+\sigma\,. \tag{3.11}$$

By virtue of condition (3.4) we can apply (A.4) to get

$$\left|\tfrac{\partial u}{\partial x_j}\right| \leqslant 2(b+\sigma)^{1/2}c_d\nu^{-1/3} \qquad \text{for all } j=1,\ldots,n,\ t\in[0,T],\ \boldsymbol{x}\in[0,\ell]^n. \tag{3.12}$$

The images of the straight line segments $\big[(0,\boldsymbol{x}_1),(0,\boldsymbol{x}_2)\big]$ and $\big[(T,\boldsymbol{x}_1),(T,\boldsymbol{x}_2)\big]$ under the map u are curves in $\mathbb{C}$ of length L_0 and L_T respectively. Using (3.12) we have

$$\max\{L_0,L_T\} \leqslant n\ell \max\left|\tfrac{\partial u}{\partial x_j}\right| \leqslant 2n\ell(b+\sigma)^{1/2}c_d\nu^{-1/3}\,. \tag{3.13}$$

By virtue of (3.10) the image of the rectangle with the vertices $(0,\boldsymbol{x}_1)$, $(0,\boldsymbol{x}_2)$, $(T,\boldsymbol{x}_2)$ and $(T,\boldsymbol{x}_1)$ under the mapping u belongs to the annulus $a-\sigma \leqslant |u| \leqslant b+\sigma$ and is contractible within the annulus. We denote the boundary of this rectangle by Γ. Consider the map $\tilde{\varphi}:\Gamma\to S^1$, $\quad\tilde{\varphi}(t,\boldsymbol{x})=\frac{|u(t,\boldsymbol{x})|}{u(t,\boldsymbol{x})}$. Here S^1 denotes the unit circle: $S^1=\{z\in\mathbb{C}:|z|=1\}$. Note that the map $\tilde{\varphi}$ is contractible[4]. The circle S^1 can be naturally identified with $\mathbb{R}/_{2\pi\mathbb{Z}}$ and we will write $S^1=\mathbb{R}/_{2\pi\mathbb{Z}}$. With this in mind the formula for the map $\tilde{\varphi}$ becomes as follows: $\tilde{\varphi}(t,\boldsymbol{x})=\arg u(t,\boldsymbol{x})$. Let $\varphi:\Gamma\to\mathbb{R}$ be the lifting of this map to the real line $\mathbb{R}$, (see, e.g. [6, p. 133]). Since the map $\tilde{\varphi}$ is contractible, the lifting φ exists and is defined up to a multiple of 2π. On the "vertical" sides of the rectangle Γ (i.e., on the sides parallel to the time axis) we have the inequality

$$u^{2p}-\tfrac{|\nu\Delta u|}{u} \leqslant \tfrac{\partial\varphi}{\partial t} \leqslant u^{2p}+\tfrac{|\nu\Delta u|}{u}\,. \tag{3.14}$$

For the "vertical" sides $\big[(0,\boldsymbol{x}_i),(T,\boldsymbol{x}_i)\big]$, $i=1,2$ we assign numbers $\theta_i=|\varphi(T,\boldsymbol{x}_i)-\varphi(0,\boldsymbol{x}_i)|$, $i=1,2$. Inequality (3.14) and bounds (3.11) imply

$$\theta_2 \geqslant T\left((b-\sigma)^{2p}-\tfrac{\nu n c_d^2\nu^{-2/3}}{b-\sigma}\right) \qquad\text{and}\qquad \theta_1 \leqslant T\left((a+\sigma)^{2p}+\tfrac{\nu n c_d^2\nu^{-2/3}}{a-\sigma}\right). \tag{3.15}$$

Similarly for "horizontal" sides $\big[(0,\boldsymbol{x}_1),(0,\boldsymbol{x}_2)\big]$ and $\big[(T,\boldsymbol{x}_1),(T,\boldsymbol{x}_2)\big]$ we assign numbers θ_0 and θ_T as follows: $\theta_\tau=|\varphi(\tau,\boldsymbol{x}_1)-\varphi(\tau,\boldsymbol{x}_2)|$, where $\tau=0$ or $\tau=T$. Since the image of Γ under the map u belongs to $\{z\in\mathbb{C}:|z|\geqslant a-\sigma\}$ we have: $\theta_0+\theta_T\leqslant\frac{L_0+L_T}{a-\sigma}$. Therefore, applying (3.13) we conclude that

$$\theta_0+\theta_T \leqslant 2\,\frac{2n\ell(b+\sigma)^{1/2}c_d\nu^{-1/3}}{a-\sigma}\,. \tag{3.16}$$

Now we employ (3.15) and (3.16) to deduce from the triangle inequality $\theta_2\leqslant\theta_1+\theta_0+\theta_T$ the following inequality:

$$c_t\nu^{-1/3}\left((b-\sigma)^{2p}-\tfrac{nc_d^2\nu^{1/3}}{b-\sigma}\right) \leqslant c_t\nu^{-1/3}\left((a+\sigma)^{2p}+\tfrac{nc_d^2\nu^{1/3}}{a-\sigma}\right)+\tfrac{4n\ell(b+\sigma)^{1/2}c_d\nu^{-1/3}}{a-\sigma}. \tag{3.17}$$

Accounting for (3.2), this inequality contradicts inequality (3.5). We set $t^\star$ to be the first moment of time when inequality (3.6) holds. Then inequality (3.10) is satisfied for $\boldsymbol{x}\in[0,\ell]^n$ and $t\in[0,t^\star]$, and we arrive at (3.7) and (3.8). ☐

Proof of theorem 1.1. Find a suitable cube in Ω and apply lemma 3.1. Since the maximum of u is bounded at $t^\star$, but the second derivative is large, we can employ the interpolation inequalities for the classical C^m-seminorms (see, e.g. [7]) to get (1.4) from (3.7) and (3.6). Finally, the continuity of the constants C_m and ν' follows since they can be constructed explicitly, see remark 3.3. ☐

4 Time-averaged Sobolev norms

In this section we prove theorem 1.2. On the space of ℓ-periodic functions

$$u(t,\boldsymbol{x})=u(t,x_1,\ldots,x_j+\ell,\ldots,x_n) \tag{4.1}$$

---

[4] Topologically $\tilde{\varphi}$ is a map from a circle into a circle. Contractibility essentially means that if the argument completes a full loop in Γ its image doesn't complete any full loop in S^1. In other words the positive movements of the image are compensated for by the negative movements.

we consider the family of (semi-) norms

$$\|u\|_k^2 = \int_{[0,\ell]^n} u\,(-\Delta)^k \bar{u}\ \mathrm{d}^n \boldsymbol{x} = \sum_{|\alpha|=k} \tbinom{|\alpha|}{\alpha} \big|D^\alpha u\big|^2_{L_2([0,\ell]^n)}\,. \tag{4.2}$$

Here $\binom{|\alpha|}{\alpha} = \binom{|\alpha|}{\alpha_1,\ldots,\alpha_n} = \frac{(\alpha_1+\cdots+\alpha_n)!}{\alpha_1!\cdots\alpha_n!}$, the coefficients in the generalized binomial equation:

$$(x_1 + x_2 + \ldots + x_n)^k = \sum_{|\alpha|=k} \tbinom{k}{\alpha} \boldsymbol{x}^\alpha\,,$$

and we note that $\sum_{|\alpha|=k} \binom{k}{\alpha} = n^k$.

Lemma 4.1. *Let u be an ℓ-periodic solution of equation (1.2). Then for any time t we have:*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_k^2 \leqslant -\nu(\operatorname{Im} z)\|u\|_{k+1}^2 + C_{k,p,n}|u|_{L_\infty}^{2p}\|u\|_k^2. \tag{4.3}$$

One can take $C_{k,p,n} = 4^{k^2}(2p+1)^k n^k$.

Proof. We multiply the equation (1.2) by $i(-\Delta)^k\bar{u}$, integrate over the period $[0,\ell]^n$ and take the real part to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_k^2 = -\nu(\operatorname{Im} z)\|u\|_{k+1}^2 + \Re\int_{[0,\ell]^n} i|u|^{2p}u(-\Delta)^k\bar{u}\ \mathrm{d}^n\boldsymbol{x}\,. \tag{4.4}$$

For brevity we will omit the domain of integration $[0,\ell]^n$ from the integral sign. To complete proof of the lemma, we need to prove the inequality

$$\Big|\int |u|^{2p}u(-\Delta)^k\bar{u}\ \mathrm{d}^n\boldsymbol{x}\Big| \leqslant C_{k,p,n}|u|_{L_\infty}^{2p}\int u(-\Delta)^k\bar{u}\,\mathrm{d}\boldsymbol{x}. \tag{4.5}$$

Integrating the left hand side of this by parts k times, we obtain

$$\int |u|^{2p}u{\textstyle\sum}\tfrac{\partial^2}{\partial x_{j_1}^2}\cdots\tfrac{\partial^2}{\partial x_{j_k}^2}\bar{u}\,\mathrm{d}\boldsymbol{x} = (-1)^k\!\int {\textstyle\sum}\tfrac{\partial}{\partial x_{j_1}}\cdots\tfrac{\partial}{\partial x_{j_k}}(u^p\bar{u}^p u)\tfrac{\partial}{\partial x_{j_1}}\cdots\tfrac{\partial}{\partial x_{j_k}}\bar{u}\,\mathrm{d}\boldsymbol{x}. \tag{4.6}$$

Both the right hand side and the left hand side contain n^k terms. Now we use the formula for differentiating the product

$$\tfrac{\partial}{\partial x_{j_1}}\cdots\tfrac{\partial}{\partial x_{j_k}}(v_1 v_2\cdots v_{2p+1}) = \sum_{\beta_1+\cdots+\beta_{2p+1}=(1,\ldots,1)}\ \prod_{l=1}^{2p+1}\Big(\tfrac{\partial}{\partial x_{j_1}},\ldots,\tfrac{\partial}{\partial x_{j_k}}\Big)^{\beta_l} v_l.$$

Here β_l are k-dimensional multi-indexes and the dimension of $(1,\ldots,1)$ is also k. The sum in the r.h.s. contains $(2p+1)^k$ terms. Hence the right-hand side of (4.6) can be represented as a sum of $(2p+1)^k n^k$ terms, each of them of the form

$$\int D^{\alpha_1}v_1\cdots D^{\alpha_{2p+1}}v_{2p+1}D^\alpha v_{2p+2}\ \mathrm{d}\boldsymbol{x}, \tag{4.7}$$

where $|\alpha| = k$ and $\alpha_1+\cdots+\alpha_{2p+1} = \alpha$; $v_l = u$ for odd l and $v_l = \bar{u}$ for even l. Using the Hölder inequality we estimate the absolute value of (4.7) as:

$$\Big|\int D^{\alpha_1}v_1\cdots D^{\alpha_{2p+1}}v_{2p+1}D^\alpha v_{2p+2}\,\mathrm{d}\boldsymbol{x}\Big| \leqslant \Big(\prod_{l=1}^{2p+1}|D^{\alpha_l}v_l|_{L_{\frac{2k}{|\alpha_l|}}}\Big)|D^\alpha v_{2p+2}|_{L_2} = \mathfrak{A}\,.$$

Using the Gagliardo-Nirenberg inequality (see [5], pp. 106-107)

$$|D^{\alpha_l}v_l|_{L_{\frac{2k}{|\alpha_l|}}} \leqslant 4^{|\alpha_l|(k-|\alpha_l|)}|v_l|_{L_\infty}^{1-\frac{|\alpha_l|}{k}}\|v_l\|_k^{\frac{|\alpha_l|}{k}}\,,$$

we obtain

$$\mathfrak{A} \leqslant \Big(\prod_{l=1}^{2p+1} 4^{|\alpha_l|(k-|\alpha_l|)} |v_l|_{L_\infty}^{1-\frac{|\alpha_l|}{k}} \|v_l\|_k^{\frac{|\alpha_l|}{k}} \Big) \|v_{2p+2}\|_k \leqslant 4^{k^2} |u|_{L_\infty}^{2p} \|u\|_k^2 \,.$$

To obtain the last inequality we used the simple inequality $|\alpha_l|(k-|\alpha_l|) \leqslant |\alpha_l|k$ and the fact that $\sum |\alpha_l| = k$. We arrive at (4.5) with $C_{k,p,n} = 4^{k^2}(2p+1)^k n^k$, so the lemma 4.1 is proven. $\square$

Using (2.9), for all sufficiently large k, we have the following estimate:

$$\|u(t^\star,\cdot)\|_k^2 \geqslant c''_k \left(\tfrac{1}{\nu}\right)^{\frac{2}{3}(k-\frac{n}{2})}, \tag{4.8}$$

where c''_k are ν-independent constants.

Suppose, that $\operatorname{Im} z \geqslant 0$; then using the fact that $|u(t)|_{L_\infty}$ remains bounded for $t \in [0, t^\star]$ (see theorem 1.1) we derive from (4.3) the following inequality:

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\|u\|_k^2 \leqslant c'\|u\|_k^2 \quad \text{for all } \ t \in [0, t^\star].$$

From this it follows that

$$\int_0^T \|u\|_k^2 \,\mathrm{d}t \geqslant \int_0^{t^\star} \|u\|_k^2 \,\mathrm{d}t \geqslant \tfrac{1}{c'} \int_0^{t^\star} \tfrac{\mathrm{d}}{\mathrm{d}t}\|u\|_k^2 \,\mathrm{d}t \geqslant \tfrac{c''_k}{c'} \left(\tfrac{1}{\nu}\right)^{\frac{2}{3}(k-\frac{n}{2})} - \tfrac{1}{c'}\|u_0\|_k^2.$$

We conclude from this that for all large enough k, and for small enough ν we have

$$\frac{1}{T}\int_0^T \|u\|_k^2 \,\mathrm{d}t \geqslant C'_k \left(\tfrac{1}{\nu}\right)^{\frac{2}{3}(k-\frac{n}{2})-\frac{1}{3}} \,. \tag{4.9}$$

Here $T = c_t \nu^{1/3}$; c_t and C'_k are ν-independent constants. The value c_t can be chosen to be 1 (see theorem 1.1). This proves theorem 1.2.

Appendix.

For a smooth function f defined on an n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/(\ell\mathbb{Z})^n$ we define the H^m Sobolev seminorm as follows:

$$\|f\|_m^2 = \int ((-\Delta)^m f)\bar{f} \,\mathrm{d}\boldsymbol{x} = \sum_{|\alpha|=m} \tbinom{|\alpha|}{\alpha} |D^\alpha f|_{L^2}^2 = \sum_{j_1,\dots,j_m=1}^{n} \left|\tfrac{\partial^m f}{\partial x_{j_1}\cdots\partial x_{j_m}}\right|_{L^2}^2.$$

Here $\binom{|\alpha|}{\alpha} = \frac{|\alpha|!}{\alpha_1!\cdots\alpha_n!}$. Note that $\sum_{|\alpha|=m}\binom{|\alpha|}{\alpha} = n^m$. In the following lemma we prove a Moser inequality with the constant $n^m 4^{m^2}$.

Lemma A1. *Let f and g be smooth functions, defined on an n-dimensional torus. Let $m \geqslant 0$. Then*

$$\|fg\|_m \leqslant n^m 4^{m^2} (|f|_{L^\infty}\|g\|_m + \|f\|_m |g|_{L^\infty}).$$

Proof. It is sufficient to estimate every term of the form $\left|\frac{\partial^m(fg)}{\partial x_{j_1}\cdots\partial x_{j_m}}\right|_{L^2}$. We note that there are n^m of them. Using the Leibniz rule for differentiating products we find that it is sufficient to estimate $\left|(D^\alpha f)(D^\beta g)\right|_{L^2}$ for any multi-indexes α and β such that $|\alpha+\beta| = m$. We note that for any multi-index γ with $|\gamma| = m$ the Leibniz expansion of $D^\gamma(fg)$ has 2^m many terms of the form $(D^\alpha f)(D^\beta g)$ with $\alpha+\beta=\gamma$. Applying the Hölder inequality we obtain

$$\left|(D^\alpha f)(D^\beta g)\right|_{L^2} \leqslant \left|(D^\alpha f)\right|_{L^{\frac{2m}{|\alpha|}}} \left|(D^\beta g)\right|_{L^{\frac{2m}{|\beta|}}} \,.$$

Next we use Gagliardo-Nirenberg inequality [5, pp.106-107]

$$\left|(D^\alpha f)\right|_{L^{\frac{2m}{|\alpha|}}} \leqslant 4^{|\alpha|(m-|\alpha|)}\, |f|_{L^\infty}^{1-\frac{|\alpha|}{m}}\, \|f\|_m^{\frac{|\alpha|}{m}}$$

to obtain

$$\left|(D^\alpha f)\right|_{L^{\frac{2m}{|\alpha|}}} \left|(D^\beta g)\right|_{L^{\frac{2m}{|\beta|}}} \leqslant 4^{m^2-|\alpha|^2-|\beta|^2} \left(|f|_{L^\infty} \|g\|_m\right)^{\frac{|\beta|}{m}} \left(|g|_{L^\infty} \|f\|_m\right)^{\frac{|\alpha|}{m}}.$$

Using the inequality $A^s B^{1-s} \leqslant sA + (1-s)B \leqslant A + B$ we arrive at

$$\left|(D^\alpha f)(D^\beta g)\right|_{L^2} \leqslant 4^{m^2-|\alpha|^2-|\beta|^2} \left(|f|_{L^\infty} \|g\|_m + |g|_{L^\infty} \|f\|_m\right).$$

Finally, using $|\alpha|^2 + |\beta|^2 \geqslant |\alpha| + |\beta| = m$, we have $2^m n^m 4^{m^2-|\alpha|^2-|\beta|^2} \leqslant 2^m n^m 4^{m^2-m} \leqslant n^m 4^{m^2}$. The lemma is proven. $\square$

Lemma A2. *For any positive δ and for any smooth function u defined on the torus $\mathbb{T}^n = \mathbb{R}^n/(\ell\mathbb{Z}^n)$ we have*

$$|u|_{L^\infty} \leqslant C_{n,\delta} \|u\|_{\frac{n}{2}-\delta}^{1/2} \|u\|_{\frac{n}{2}+\delta}^{1/2} + \left|\langle u\rangle\right|, \tag{A.1}$$

where $\langle u\rangle = \frac{1}{\operatorname{Vol}\mathbb{T}^n}\int_{\mathbb{T}^n} u(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ is the mean value of the function u. The constant $C_{n,\delta}$ does not depend on u or the size ℓ of the torus $\mathbb{T}^n$.

Proof. Consider the Fourier decomposition $u(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^n} \hat{u}_{\boldsymbol{k}} \exp(\frac{2\pi i\boldsymbol{k}\cdot\boldsymbol{x}}{\ell})$. Without loss of generality we assume $\langle u\rangle = \hat{u}_{\boldsymbol{0}} = 0$. Fix any positive real number r we have

$$\begin{aligned}
|u|_{L^\infty} \leqslant \sum_{\boldsymbol{k}\in\mathbb{Z}^n\setminus\{\boldsymbol{0}\}} |\hat{u}_{\boldsymbol{k}}| &= \sum_{0<|\boldsymbol{k}|\leqslant r} |\hat{u}_{\boldsymbol{k}}| + \sum_{|\boldsymbol{k}|>r} |\hat{u}_{\boldsymbol{k}}| = \\
&= \sum_{0<|\boldsymbol{k}|\leqslant r} |\boldsymbol{k}|^{-\frac{n}{2}+\delta} |\boldsymbol{k}|^{\frac{n}{2}-\delta} |\hat{u}_{\boldsymbol{k}}| + \sum_{|\boldsymbol{k}|>r} |\boldsymbol{k}|^{-\frac{n}{2}-\delta} |\boldsymbol{k}|^{\frac{n}{2}+\delta} |\hat{u}_{\boldsymbol{k}}| \leqslant \\
&\leqslant \Big(\sum_{0<|\boldsymbol{k}|\leqslant r} |\boldsymbol{k}|^{-n+2\delta}\Big)^{\frac{1}{2}} \Big(\sum_{0<|\boldsymbol{k}|\leqslant r} |\boldsymbol{k}|^{n-2\delta} |\hat{u}_{\boldsymbol{k}}|^2\Big)^{\frac{1}{2}} + \Big(\sum_{|\boldsymbol{k}|>r} |\boldsymbol{k}|^{-n-2\delta}\Big)^{\frac{1}{2}} \Big(\sum_{|\boldsymbol{k}|>r} |\boldsymbol{k}|^{n+2\delta} |\hat{u}_{\boldsymbol{k}}|^2\Big)^{\frac{1}{2}}.
\end{aligned}$$

Denote $\psi_1(r) = \big(\sum_{0<|\boldsymbol{k}|\leqslant r} |\boldsymbol{k}|^{-n+2\delta}\big)^{\frac{1}{2}}$ and $\psi_2(r) = \big(\sum_{|\boldsymbol{k}|>r} |\boldsymbol{k}|^{-n-2\delta}\big)^{\frac{1}{2}}$. Then for any $r > 0$ we have

$$|u|_{L^\infty} \leqslant (2\pi)^{-n/2} \left(\psi_1(r) \|u\|_{\frac{n}{2}-\delta} \left(\tfrac{2\pi}{\ell}\right)^{\delta} + \psi_2(r) \|u\|_{\frac{n}{2}+\delta} \left(\tfrac{2\pi}{\ell}\right)^{-\delta}\right).$$

We claim that there exist two positive constants $C'_{n,\delta}$ and $C''_{n,\delta}$ such that

$$\psi_1(r) \leqslant C'_{n,\delta} r^{\delta} \quad \text{and} \quad \psi_2(r) \leqslant C''_{n,\delta} r^{-\delta}. \tag{A.2}$$

To prove this claim we first, using integral bounds for sums, establish (A.2) asymptotically as $r \to \infty$, i.e. for $r > R$ for large enough R. Second, we note that ψ_2 is bounded and ψ_1 is equal to zero for $r < 1$. Thus we obtain

$$|u|_{L^\infty} \leqslant (2\pi)^{-n/2} \left(C' \|u\|_{\frac{n}{2}-\delta} \left(r\tfrac{2\pi}{\ell}\right)^{\delta} + C'' \|u\|_{\frac{n}{2}+\delta} \left(r\tfrac{2\pi}{\ell}\right)^{-\delta}\right).$$

Now taking r to satisfy $\left(r\frac{2\pi}{\ell}\right)^{\delta} = \sqrt{C'' \|u\|_{\frac{n}{2}+\delta} / (C' \|u\|_{\frac{n}{2}-\delta})}$ we arrive at (A.1). $\square$

Lemma A3. *For fixed f the function $m \mapsto \|f\|_m$ is* log-*convex.*

Proof. Consider the Fourier representation $f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^n} \hat{f}_{\boldsymbol{k}} \exp(\frac{2\pi i\boldsymbol{k}\cdot\boldsymbol{x}}{\ell})$, then

$$\|f\|_m^2 = \ell^n \left(\tfrac{2\pi}{\ell}\right)^{2m} \sum_{\boldsymbol{k}\in\mathbb{Z}^n} |\boldsymbol{k}|^{2m} |\hat{f}_{\boldsymbol{k}}|^2.$$

It is sufficient to prove that the map $m \mapsto \sum |\boldsymbol{k}|^{2m} |\hat{f}_{\boldsymbol{k}}|^2$ is log-convex. This follows from the Hölder inequality,

$$\sum |\boldsymbol{k}|^{2(sm_1+(1-s)m_2)} |\hat{f}_{\boldsymbol{k}}|^2 = \sum (|\boldsymbol{k}|^{2m_1} |\hat{f}_{\boldsymbol{k}}|^2)^s (|\boldsymbol{k}|^{2m_2} |\hat{f}_{\boldsymbol{k}}|^2)^{1-s} \leqslant \big(\sum |\boldsymbol{k}|^{2m_1} |\hat{f}_{\boldsymbol{k}}|^2\big)^s \big(\sum |\boldsymbol{k}|^{2m_2} |\hat{f}_{\boldsymbol{k}}|^2\big)^{1-s}.$$

Here $s \in [0,1]$. $\square$

Lemma A4. *Let u be a C^2-smooth complex valued function defined on the interval $[0,\ell]$. Let M_0 and M_2 be positive real numbers such that $[u]_0 \leqslant M_0$ and $[u]_2 \leqslant M_2$. Then we have*

$$[u]_1 \leqslant \tfrac{2M_0}{\ell} + \tfrac{\ell M_2}{2} \qquad \text{if} \quad \ell^2 \leqslant \tfrac{4M_0}{M_2}\,, \tag{A.3}$$

$$[u]_1 \leqslant 2M_0^{1/2}M_2^{1/2} \qquad \text{if} \quad \ell^2 \geqslant \tfrac{4M_0}{M_2}\,. \tag{A.4}$$

Proof. We can reduce the lemma to the real case by considering the function $v = \mathrm{Re}\{e^{i\theta}u\}$ for appropriate θ. The lemma is scale invariant, so without loss of generality, we consider $M_0 = M_2 = 1$. The proof then follows by consideration of piecewise quadratic functions and the extreme case $\ell = 2$ and $u(x) = x^2/2 - 1$. $\square$

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The following theorem have been removed from the final publication.

5 Removed theorem.

To illustrate the explicit constants mentioned in the introduction, we consider the case of functions u that satisfy periodic boundary conditions, i.e., for any $j = 1, \dots, n$ we have:

$$u(t, \boldsymbol{x}) = u(t, x_1, \dots, x_j, \dots, x_n) = u(t, x_1, \dots, x_j + \ell, \dots, x_n)\,. \tag{5.1}$$

We will also use a technical assumption that u takes the value zero at some point initially. This assumption is satisfied for example, in the odd-periodic case.

Theorem 5.1. *Let u be a periodic solution of equation (1.2), such that u_0 takes value zero. Let*

$$c_d = \tfrac{(4^{2p}-3^{2p})^{1/3}}{2n^{2/3}\ell^{1/3}5^{2p/3+1/2}6^{1/6}}|u_0|^{2p/3+1/2} \qquad \textit{and} \qquad c_t = \tfrac{4(6n\ell^2 5^{4p})^{1/3}}{(4^{2p}-3^{2p})^{2/3}|u_0|^{4p/3}}\,. \tag{5.2}$$

Suppose that

$$\nu < \tfrac{(4^{2p}-3^{2p})\ell^2}{2304n^2 5^{2p}}|u_0|^{2p}; \tag{5.3}$$

then there exists $t^\star \in [0, T]$, where $T = c_t \nu^{-1/3}$ such that for any $k \geqslant 2$ we have

$$\max_{j=1,\dots,n} \sup_{\boldsymbol{x}\in[0,\ell]^n} \left|\frac{\partial^k u}{\partial x_j^k}(t^\star, \boldsymbol{x})\right| \geqslant \frac{1}{\sqrt{3}} \frac{c_d^k \nu^{-k/3}}{\left(\frac{6}{5}\,|u_0|_{L_\infty}\right)^{\frac{k-2}{2}}}\,. \tag{5.4}$$

$$\sup_{\substack{\boldsymbol{x}\in[0,\ell]^n \\ t\in[0,t^\star]}} \big|u(t^\star, \boldsymbol{x})\big| \leqslant \frac{6}{5}|u_0|\,. \tag{5.5}$$

Proof. By continuity, $|u_0(\boldsymbol{x})|$ takes all values between zero and $\max|u_0|$. We set $b = |u_0|$, $a = \frac{2}{5}\,|u_0|$, $\sigma = \frac{1}{5}\,|u_0|$ and use lemma 3.1. The estimate (5.5) follows immediately. The estimate (5.4) is obtained by the extrapolation of (3.6) and (5.5), using the interpolation inequality $|f^{(l)}|_{L_\infty(\mathbb{R})} \leqslant C_{l,k}|f|^{1-l/k}_{L_\infty(\mathbb{R})}|f^{(k)}|^{l/k}_{L_\infty(\mathbb{R})}$ (see [7]) with $l = 2$ and

$$f(x) = u(t^\star, \tilde{x}_1, \dots, \tilde{x}_{j-1}, x, \tilde{x}_{j+1}, \dots, \tilde{x}_n),$$

where j is the index of the derivative and $(\tilde{x}_1, \dots, \tilde{x}_n)$ is the point where supremum in (3.6) is achieved. We also used the inequality $C_{2,k}^{k/2} \leqslant \sqrt{3}$ which follows from the explicit values for the constants $C_{l,k}$ given in [7] . ☐