Spectral Properties of Solutions of Burgers Equation with Small Dissipation

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Functional Analysis and Its Applications. Vol. **35.**, no 1., 2001.

1 Introduction

This present paper concerns the initial value problem for the one dimensional ($\dim x = 1$) parabolic equation of Burgers type:

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}f(u) = \delta u_{xx} \ , \tag{1.1}$$

with the initial state

$$u(0,x) = u_0(x) \ , \tag{1.2}$$

and satisfying the periodical boundary conditions

$$u(t, x \pm \ell) = u(t,x) \ . \tag{1.3}$$

Here δ is a small (< 1) positive parameter, the initial state is a C^∞-smooth and ℓ-periodic function $u_0(\cdot)$. The function $f : \mathbb{R} \to \mathbb{R}$ is C^∞-smooth and uniformly convex ($f'' \geqslant \sigma > 0$).

The classical theory of parabolic equations (see [1]) states that problems (1.1), (1.2), (1.3) has a unique solution $u = u^\delta(t,x)$, for every $\delta > 0$ which is a C^∞-smooth. It has also been known that while $\delta \to +0$ the solution of the problem (1.1), (1.2), (1.3) converges to the so called (weak) entropy solution of quasilinear equation

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \ . \tag{1.1$'$}$$

See [2], where this fact has been studied under the most general assumptions. Moreover in [3] it has been shown that rate of convergence in the $L_1(\mathbb{R})$ norm is at least $Const \cdot \sqrt{\delta}$. See Appendix for the proof of this fact for our situation.

We remark that equation (1.1$'$) is, in general, not solvable even in the class of continuous functions. For this reason it is normal to consider weak solutions in the sense of corresponding integral identity. However, this approach leads to loss of uniqueness. Imposing an entropy condition ensures uniqueness and the resulting unique solution is the most natural from the physical point of view.

We shall study behaviour of the H^m-Sobolev seminorms

$$\|u\|_m^2 = \int \left| \frac{\partial^m u}{\partial x^m} \right|^2 dx$$

as $\delta \to 0$. In this paper $\int ...dx$ (i.e., we do not specify the limits) denotes $\int_0^\ell ...dx = \int_{x_0}^{x_0+\ell} ...dx$, i.e., the integral over the period or, equivalently, the integral over the circle $S = {}^{\mathbb{R}}/_{\ell\mathbb{Z}}$. Henceforth we will omit the prefix semi- from the word seminorm.

The H^m-Sobolev norm is a function, which depends on t. We consider the average of this function over [0,T]:

$$\left\langle \|u\|_m^2 \right\rangle_{[0,T]} = \frac{1}{T}\int_0^T \int \left| \frac{\partial^m u}{\partial x^m} \right|^2 dx dt \ .$$

We shall show that for any $m \geqslant 1$, δ is small enough and $t \in \mathbb{R}_+$ we have that

$$\|u\|_m \leqslant C_m \left(\frac{1}{\delta}\right)^{m-1/2} . \tag{1.4}$$

On the other hand, there exists a number T, which depends on the initial condition of problem (1.1), (1.2), (1.3) (see formula (3.2)) such that

$$\left\langle \|u\|_m^2 \right\rangle_{[0,T]}^{1/2} \geqslant c_m \left(\frac{1}{\delta}\right)^{m-1/2} . \tag{1.5}$$

Here the constants c_m and C_m does not depend on δ, and could be explicitly expressed in the terms of the C^k-norm ($k \leqslant m+1$) of the initial state of problem (1.1), (1.2), (1.3).

The H^m-Sobolev norm can be expressed in terms of Fourier coefficients:

$$\|u\|_m^2 = \left(\frac{2\pi}{\ell}\right)^{2m} \sum_{s=-\infty}^{+\infty} s^{2m} |\hat{u}_s|^2 ,$$

where $\hat{u}_s = \frac{1}{\sqrt{\ell}} \int u(x) e^{-\frac{2\pi i s x}{\ell}} dx$. Together with (1.4) and (1.5) this relation makes it possible to get some information about the behaviour of the Fourier coefficients of the solution as $\delta \to +0$. Let $\theta > 0$ be a fixed number. By $\hat{A}^2_{s,\theta}$ we denote the average of $|\hat{u}_k|^2(t)$ over $t \in [0;T]$ and $\left(\frac{1}{\delta}\right)^{s-\theta} < |k| \leqslant \left(\frac{1}{\delta}\right)^{s+\theta}$, i.e.,

$$\hat{A}^2_{s,\theta} = \frac{1}{T} \int_0^T \frac{\sum\limits_{\delta^{-s+\theta} < |k| \leqslant \delta^{-s-\theta}} |\hat{u}_k|^2(t)}{\sum\limits_{\delta^{-s+\theta} < |k| \leqslant \delta^{-s-\theta}} 1} dt ,$$

here $s > 0$. Then for all small enough δ the values $\hat{A}^2_{s,\theta}$ satisfy to the following estimates:

$$\hat{A}^2_{s,\theta} < \frac{\ell}{\pi^2} \left|\frac{du_0}{dx}\right|^2_{L_1} \delta^{2s} \qquad \text{for } s > 0 \quad , \tag{1.6}$$

$$\hat{A}^2_{s,\theta} < \left(\frac{\ell}{2\pi}\right)^{2m} C_m^2\, \delta^{2+2m(s-1-\theta)} \qquad \forall m > 0 \qquad \text{for } s > 1+\theta , \tag{1.7}$$

$$\hat{A}^2_{1,\theta} > \left(\frac{\ell}{2\pi}\right)^2 \frac{c_1^2}{4}\, \delta^{2+2\theta} \qquad \text{for } s = 1 \quad . \tag{1.8}$$

This article is devoted to the proving of estimates (1.4) — (1.8) for the problem (1.1), (1.2), (1.3).

We note that relations (1.6) — (1.8) are closely related to the so called Kolmogorov asymptotics for freely decaying turbulence (i.e., unforced turbulence), described by Burgers equation. In particularly it follows from (1.6) — (1.8) that the Kolmogorov dissipation scale of solutions of Burgers equation (1.1) is equal to δ and the exponent in the spectral law is equal to two (see [8], cf. [9],[10]).

The following interpolation inequalities will be used subsequently:

$$\|u\|_k \leqslant \|u\|_0^{1-\frac{k}{m}} \|u\|_m^{\frac{k}{m}} \qquad (0 \leqslant k \leqslant m) , \tag{1.9}$$

$$|u|_k \leqslant C_{mk} |u|_0^{1-\frac{k}{m}} |u|_m^{\frac{k}{m}} \qquad (0 \leqslant k \leqslant m) . \tag{1.10}$$

Here $|u|_k = \max_{x\in S}\left|\frac{d^k u}{dx^k}\right|$. The proof of inequality (1.9) is simple:

$$\left(\frac{\ell}{2\pi}\right)^{2k}\|u\|_k^2 = \sum\left(|\hat{u}_s|^2\right)^{\frac{m-k}{m}}\cdot\left(|\hat{u}_s|^2\cdot s^{2m}\right)^{\frac{k}{m}} \leqslant$$

$$\leqslant \left(\sum|\hat{u}_s|^2\right)^{\frac{m-k}{m}}\cdot\left(\sum|\hat{u}_s|^2\cdot s^{2m}\right)^{\frac{k}{m}} = \|u\|_0^{1-\frac{k}{m}}\cdot\left(\left(\frac{\ell}{(2\pi)}\right)^{2m}\|u\|_m^2\right)^{\frac{k}{m}} .$$

The proof of inequality (1.10) is not trivial. It can be found in [4]. It is also shown in [4] that the optimal values of the constants C_{mk} are:

$$C_{mk} = \frac{K_{m-k}}{K_m^{1-\frac{k}{m}}}\ , \text{ where } \qquad K_i = \frac{4}{\pi}\left(1+\frac{(-1)^{i+1}}{3^{i+1}}+\frac{1}{5^{i+1}}+\frac{(-1)^{i+1}}{7^{i+1}}+...\right) .$$

Further on, we shall use the following simple fact:

$$(C_{m1})^m \leqslant 2\ . \tag{1.11}$$

Besides this we shall use the Gagliardo-Nirenberg inequality for the periodic functions:

$$\left|\frac{\partial^\alpha u}{\partial x^\alpha}\right|_{L_{\frac{2m}{\alpha}}} \leqslant 4^{\alpha(m-\alpha)}|u|_{L_\infty}^{1-\frac{\alpha}{m}}\left|\frac{\partial^m u}{\partial x^m}\right|_{L_2}^{\alpha/m} , \tag{1.12}$$

where α is an integer between 0 and m (see [5], pp. 106-107).

We denote $\frac{\partial^m u}{\partial x^m}$ by $u^{(m)}$, similarly $u_x = \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, and $u_t = \frac{\partial u}{\partial t}$. Unless otherwise stated we let L_p denote the space $L_p(S)$.

The author would like to acknowledge Professor S. B. Kuksin, who originally formulated the problem, for his attention to this research.

2 Upper estimates

Let u_+ and u_- be maximum and minimum value of the initial state $u_0 = u_0(x)$. The oscillation of the solution at time t is $\operatorname{osc}_x u(t,x) = \max_x u(t,x) - \min_x u(t,x)$. We define the constant D to be

$$D = \frac{1}{2}\operatorname{osc}_x u\Big|_{t=0} = \frac{u_+ - u_-}{2} . \tag{2.1}$$

Then, by virtue of maximum principle, we have $u_- \leqslant u(t,x) \leqslant u_+$ and

$$\frac{1}{2}\operatorname{osc}_x u(t,x) \leqslant D \quad \text{ for all } t . \tag{2.2}$$

Multiplying equation (1.1) by u and integrating over period we get:

$$\int uu_t dx = -\delta\int u_x^2 dx , \tag{2.3}$$

or

$$\delta\int u_x^2 dx = -\int uu_t dx \leqslant \frac{\operatorname{osc}_x u(t,x)}{2}\,|u_t|_{L_1} \leqslant D\cdot\left|u_t\big|_{t=0}\right|_{L_1} .$$

In the last inequality we have used inequality (2.2) and the fact that $|u_t|_{L_1} = \int \left|\frac{\partial u}{\partial t}(t,x)\right| dx$ does not increase with t, see lemma A in the Appendix. We denote constant C_1 as

$$C_1 = \sqrt{D \cdot \left(\int \left|\frac{\partial f(u_0)}{\partial x}\right| dx + \int \left|\frac{\partial^2 u_0}{\partial x^2}\right| dx\right)}\ . \tag{2.4}$$

Expressing $u_t\big|_{t=0}$ via equation (1.1) we have

$$\|u^\delta\|_1 \leqslant C_1 \delta^{-1/2} \tag{2.5}$$

for all $t \geqslant 0$ and $\delta \in (0,1)$.

Differentiating equation (1.1) $m-1$ times with respect to x, multiplying by $u^{(m+1)}$, and integrating over period we obtain:

$$\frac{1}{2}\frac{d}{dt}\int \left(\frac{\partial^m u}{\partial x^m}\right)^2 dx = -\delta \int \left(\frac{\partial^{m+1} u}{\partial x^{m+1}}\right)^2 dx + \int \frac{\partial^{m+1} u}{\partial x^{m+1}} \cdot \frac{\partial^m f(u)}{\partial x^m} dx\ . \tag{2.6}$$

To prove the main estimates of this section we need the following technical inequality.

Lemma. *For all $m \geqslant 1$ we have:*

$$\int \left|\frac{\partial^{m+1} u}{\partial x^{m+1}}\right| \cdot \left|\frac{\partial^m f(u)}{\partial x^m}\right| dx \leqslant B_m \cdot \|u\|_m \|u\|_{m+1}\ , \tag{2.7}$$

where

$$B_m = Q_m |f(\cdot)|_{C^m[u_-,u_+]} \max\left(1, |u_0|_{L_\infty}^{m-1}\right)\ , \tag{2.8}$$

and Q_m depends on m only.

Proof. By induction on m it is easy to prove that

$$\frac{\partial^m f(u)}{\partial x^m} = \sum_{k=1}^m \frac{d^k f(u)}{du^k} \left(\sum_{\substack{\alpha_1+\ldots+\alpha_k=m \\ 1\leqslant\alpha_1\leqslant\ldots\leqslant\alpha_k}} A^{k,m}_{\alpha_1,\ldots,\alpha_k} u^{(\alpha_1)} u^{(\alpha_2)} \ldots u^{(\alpha_k)} \right)\ ,$$

where the constants $A^{k,m}_{\alpha_1,\ldots,\alpha_k}$ are nonnegative integers. It follows from this that

$$\int \left|\frac{\partial^{m+1} u}{\partial x^{m+1}}\right| \cdot \left|\frac{\partial^m f(u)}{\partial x^m}\right| dx \leqslant$$

$$\leqslant \sum_{k=1}^m \sum_{\substack{\alpha_1+\ldots+\alpha_k=m \\ 1\leqslant\alpha_1\leqslant\ldots\leqslant\alpha_k}} A^{k,m}_{\alpha_1,\ldots,\alpha_k} \left|\frac{d^k f(u)}{du^k}\right| \int |u^{(m+1)}||u^{(\alpha_1)}|\ldots|u^{(\alpha_k)}| dx \leqslant$$

we use Hölder's inequality (note: $\frac{1}{2} + \frac{\alpha_1}{2m} + \ldots + \frac{\alpha_k}{2m} = 1$):

$$\leqslant \sum_{k=1}^m \sum_{\substack{\alpha_1+\ldots+\alpha_k=m \\ 1\leqslant\alpha_1\leqslant\ldots\leqslant\alpha_k}} A^{k,m}_{\alpha_1,\ldots,\alpha_k} \left|\frac{d^k f(u)}{du^k}\right|_{L_\infty} |u^{(m+1)}|_{L_2} |u^{(\alpha_1)}|_{L_{\frac{2m}{\alpha_1}}} \ldots |u^{(\alpha_k)}|_{L_{\frac{2m}{\alpha_k}}} \leqslant$$

we use inequality (1.12), noting that $\prod_{i=1}^k 4^{\alpha_i(m-\alpha_i)} \leqslant 4^{m^2}$:

$$\leqslant \sum_{k=1}^m \sum_{\substack{\alpha_1+\ldots+\alpha_k=m \\ 1\leqslant\alpha_1\leqslant\ldots\leqslant\alpha_k}} A^{k,m}_{\alpha_1,\ldots,\alpha_k} \left|\frac{d^k f(u)}{du^k}\right|_{L_\infty} |u^{(m+1)}|_{L_2}\, 4^{m^2} |u|^{k-1}_{L_\infty} |u^{(m)}|_{L_2} =$$

$$= 4^{m^2}\left(\sum_{k=1}^{m}\sum_{\substack{\alpha_1+\ldots+\alpha_k=m\\ 1\leqslant\alpha_1\leqslant\ldots\leqslant\alpha_k}} A^{k,m}_{\alpha_1,\ldots,\alpha_k}\left|\frac{d^k f(u)}{du^k}\right|_{L_\infty}|u|^{k-1}_{L_\infty}\right)\|u\|_m\,\|u\|_{m+1}\ .$$

It follows from the maximum principle that $|u|_{L_\infty}\leqslant|u_0|_{L_\infty}$. Thus the lemma is proved. □

Combining (2.6) and (2.7) we obtain:

$$\frac{1}{2}\frac{d}{dt}\|u\|_m^2\leqslant-\delta\|u\|^2_{m+1}+B_m\|u\|_m\|u\|_{m+1}=\|u\|_{m+1}\left(-\delta\|u\|_{m+1}+B_m\|u\|_m\right). \tag{2.9}$$

Theorem. *Suppose* $m\in\mathbb{N}$,

$$\delta\leqslant\delta_0'=B_m\cdot\left(\frac{C_1}{\sqrt{B_m}\,\|u_0\|_m}\right)^{\frac{1}{m-1/2}}\ , \tag{2.10}$$

and $t\geqslant 0$*; then*

$$\|u(t,\cdot)\|_m\leqslant B_m^{m-1}C_1\delta^{-m+\frac{1}{2}}\ . \tag{2.11}$$

Here C_1 *and* B_m *are defined in (2.4) and (2.8) respectively.*

Proof. Substituting u_x for u in (1.9), we obtain:

$$\|u\|_m\leqslant\|u\|_1^{\frac{1}{m}}\cdot\|u\|_{m+1}^{\frac{m-1}{m}}\ . \tag{2.12}$$

We rearrange this as $\|u\|_{m+1}\geqslant\|u\|_m\left(\frac{\|u\|_m}{\|u\|_1}\right)^{\frac{1}{m-1}}$. Combining (2.9) and the last inequality we have:

$$\frac{1}{2}\frac{d}{dt}\|u\|_m^2\leqslant\|u\|_{m+1}\|u\|_m\left(-\delta\left(\frac{\|u\|_m}{\|u\|_1}\right)^{\frac{1}{m-1}}+B_m\right)\ .$$

Hence if $\|u\|_m>\left(\frac{B_m}{\delta}\right)^{m-1}\|u\|_1$ then $\|u\|_m$ is decreasing. Combining this with (2.5) we have:

$$\text{if}\quad\|u(t,\cdot)\|_m>B_m^{m-1}C_1\delta^{-m+\frac{1}{2}},\quad\text{then}\quad\|u\|_m\quad\text{is decreasing.} \tag{2.13}$$

Condition (2.10) means that (2.11) holds at $t=0$. Combining this with (2.13) we conclude that $\|u\|_m$ can never surpass the r.h.s. of (2.11). □

3 Lower estimates

In this section we shall use the explicit formula for the weak entropy solution $u^0(t,x)$ of the quasilinear equation (1.1′), obtained by Lax (see [6]).

Let the function g be the Legendre transformation of the convex, smooth function f in (1.1′). Let $y_{min}=y_{min}(t,x)$ be the point of the global minimum of the auxiliary function $\psi_{t,x}(y)=\int_0^y u_0(\eta)d\eta+t\cdot g\left(\frac{x-y}{t}\right)$. The point y_{min} is unique for almost all t and x. Then $u^0(t,x)=g'\left(\frac{x-y_{min}(t,x)}{t}\right)$.

Let $M=\frac{1}{\ell}\int u_0dx$ denote the mean value of the initial state. It then follows from the explicit formula for the solution of equation (1.1′) that in our (ℓ-periodic) case we have

$$\sup_x|u^0(t,x)-M|\leqslant\frac{\ell}{t\cdot\sigma}\ . \tag{3.1}$$

Indeed, let y_1 be the point of the minimum of the function $t\cdot g\left(\frac{x-y}{t}\right)+My$, i.e., $g'\left(\frac{x-y_1}{t}\right)=M$. Since the functions f' and g' are inverses of each other, we have $\frac{x-y_1}{t}=f'(M)$. It follows from the ℓ-periodicity of the function $\int_0^y u_0(\eta)d\eta - My$ that $|y_{min}-y_1|<\ell$. Thus we have $\frac{x-y_{min}}{t}=f'(M)+\kappa$, where $|\kappa|\leqslant\frac{\ell}{t}$. The result is

$$|u^0(t,x)-M| = |g'\left(f'(M)+\kappa\right)-g'\left(f'(M)\right)| \leqslant \sup|g''|\cdot|\kappa| \leqslant \frac{1}{\sigma}\frac{\ell}{t}\ ,$$

as (3.1) states. It now follows that $\int\left(u^0(t,x)-M\right)^2dx \leqslant \frac{\ell^3}{t^2\cdot\sigma^2}$. Hence at the time

$$T=\frac{2\ell^{3/2}}{\sigma\,|u_0-M|_{L_2}} \tag{3.2}$$

we have

$$\int\left(u^0(T,x)-M\right)^2dx \leqslant \frac{1}{4}|u_0-M|^2_{L_2}\ . \tag{3.3}$$

In what follows we assume that $M=0$, in order to simplify formulae. This assumption has been made without loss of generality because we can make the substitution $u\to u-M$, $f(\cdot)\to f(\cdot+M)$.

Theorem. *Suppose* $m\in\mathbb{N}$ *and* $\delta<\delta_0''=\dfrac{1}{3072}\dfrac{|u_0|^4_{L_2}}{T\cdot|u_0|^2_{L_\infty}\cdot\left(\int\left|\frac{du_0}{dx}\right|dx\right)^2}$; *then we have*

$$\left(\int_0^T \|u(t,\cdot)\|_m^2dt\right)^{1/2} \geqslant c_m\left(\frac{1}{\delta}\right)^{m-1/2}\ . \tag{3.4}$$

Here T *is defined in (3.2) and* c_m *does not depend on* δ.

Proof. We rearrange (2.3) as

$$\delta\int\left(u_x^\delta(t,x)\right)^2dx = -\frac{1}{2}\frac{d}{dt}\int\left(u^\delta(t,x)\right)^2dx\ .$$

We integrate this over $[0,T]$ with respect to t

$$\int_0^T\delta\|u^\delta\|_1^2dt = \frac{1}{2}\int\left(u^\delta(0,x)\right)^2dx - \frac{1}{2}\int\left(u^\delta(T,x)\right)^2dx\ . \tag{3.5}$$

We shall have proved inequality (3.4) for $m=1$ if we can show that the last term in (3.5) is small. Suppose $\delta<\delta_0''$; then using inequality (A.6) we have:

$$\begin{aligned}\int\left(u^\delta(T,x)\right)^2dx-\int\left(u^0(T,x)\right)^2dx &\leqslant\\ \leqslant\int\left|u^\delta(T,x)-u^0(T,x)\right|\cdot\left|u^\delta(T,x)+u^0(T,x)\right|dx &\leqslant\\ \leqslant\frac{4}{\sqrt{3}}\sqrt{T\delta}\cdot\int\left|\frac{du_0}{dx}\right|dx\cdot 2\,|u_0|_{L_\infty} \leqslant \frac{1}{12}\,|u_0|^2_{L_2}&\ .\end{aligned}$$

Combining this with (3.5) and (3.3) we have

$$\int_0^T\delta\|u^\delta\|_1^2dt \geqslant \frac{1}{2}|u_0|^2_{L_2}-\frac{1}{2}\left(\frac{1}{4}|u_0|^2_{L_2}+\frac{1}{12}|u_0|^2_{L_2}\right)=\frac{1}{3}|u_0|^2_{L_2}\ ,$$

or

$$\frac{1}{T}\int\limits_0^T \|u^\delta\|_1^2 dt \geqslant \frac{|u_0|^2_{L_2}}{3T\delta}\ . \tag{3.6}$$

So inequality (3.4) is proved for $m=1$ with the constant $c_1 = \frac{|u_0|_{L_2}}{\sqrt{3T}}$.

It follows from (2.2) and the interpolation inequality (1.10) with $k=1$ that $|u_x|_{L_\infty} \leqslant C_{m1}\cdot D^{\frac{m-1}{m}}\cdot |u^{(m)}|^{\frac{1}{m}}_{L_\infty}$, where D is defined in (2.1). Using (1.11), we obtain $\left|u^{(m)}\right|_{L_\infty} \geqslant \frac{\left(|u_x|_{L_\infty}\right)^m}{2D^{m-1}}$. Using inequality (A.4) we get $\left|\frac{\partial u^\delta}{\partial x}\right|_{L_\infty} \geqslant \frac{\|u^\delta\|_1^2}{\left|\frac{du_0}{dx}\right|_{L_1}}$. Thus we have

$$\left|u^{(m)}\right|_{L_\infty} \geqslant \frac{\|u^\delta\|_1^{2m}}{2D^{m-1}\left|\frac{du_0}{dx}\right|^m_{L_1}}\ . \tag{3.7}$$

Lemma. *Let v be a periodic function that achieves the value 0 then*

$$|v|^2_{L_\infty} \leqslant |v|_{L_2}\cdot|v_x|_{L_2}\ .$$

Proof. For any y we have

$$v^2(y) = \int\limits_{v^{-1}(0)}^{y} \frac{d}{dx}v^2(x)dx \leqslant \frac{1}{2}\int\left|\frac{d}{dx}v^2(x)\right|dx = \int|v(x)||v'(x)|dx \leqslant |v|_{L_2}|v_x|_{L_2}\ .$$

□

We apply this lemma with $v=\frac{\partial^m u}{\partial x^m}$ and using inequality (2.12) we get

$$\left|u^{(m)}\right|_{L_\infty} \leqslant \|u\|_m^{1/2}\cdot\|u\|^{1/2}_{m+1} \leqslant \|u\|_1^{\frac{1}{2m}}\cdot\|u\|_{m+1}^{\frac{2m-1}{2m}}\ .$$

Hence

$$\|u\|_{m+1} \geqslant \left(\frac{|u^{(m)}|^{2m}_{L_\infty}}{\|u\|_1}\right)^{\frac{1}{2m-1}} \overset{(3.7)}{\geqslant} \frac{\|u\|_1^{2m+1}}{\left(2D^{m-1}\left|\frac{du_0}{dx}\right|^m_{L_1}\right)^{\frac{2m}{2m-1}}}\ .$$

Let's square the left and right hand sides of this inequality. Using the inequality $\frac{1}{T}\int_0^T\|u\|_1^{4m+2}dt \geqslant \left(\frac{1}{T}\int_0^T\|u\|_1^2dt\right)^{2m+1}$, we take average over $t\in[0,T]$:

$$\frac{1}{T}\int_0^T\|u\|^2_{m+1}dt \geqslant \frac{\left(\frac{1}{T}\int_0^T\|u\|_1^2dt\right)^{2m+1}}{\left(2D^{m-1}\left|\frac{du_0}{dx}\right|^m_{L_1}\right)^{\frac{4m}{2m-1}}} \overset{(3.6)}{\geqslant} \frac{\left(\frac{|u_0|^2_{L_2}}{3T\delta}\right)^{2m+1}}{\left(2D^{m-1}\left|\frac{du_0}{dx}\right|^m_{L_1}\right)^{\frac{4m}{2m-1}}}\ .$$

Thus inequality (3.4) is proved for all $m\geqslant 1$. □

It is interesting to compare inequality (3.4) for the case $m=1$ with the upper estimate for the averaged L_1-norm (which follows from inequality (A.4)):

$$\left\langle\int u'(x)^2dx\right\rangle^{1/2} \geqslant c_1\delta^{-1/2}\ , \qquad \left\langle\int|u'(x)|dx\right\rangle \leqslant Const\ .$$

Broadly speaking this means that the graph of the gradient $u'(x)$ is a curve with high and infrequent peaks. This behaviour is closely related with the phenomenon of intermittency from the theory of turbulence, see [8].

4 Behaviour of the Fourier coefficients

In this section we prove estimates (1.6) — (1.8).

The Fourier coefficients of a function, defined on the circle S are the coefficients of its expansion in the orthonormal basis $e_k(x) = \frac{1}{\sqrt{\ell}} \exp\left(\frac{2\pi i k x}{\ell}\right)$, $k \in \mathbb{Z}$. The expansion of $u(x)$ in this basis has the form

$$u(x) = \sum_{k=-\infty}^{+\infty} \hat{u}_k e_k(x)\,, \quad \text{where } \hat{u}_k = \frac{1}{\sqrt{\ell}} \int u(x) \exp\left(\frac{-2\pi i k x}{\ell}\right) dx$$

The H^m-Sobolev norm can be written as

$$\|u\|_m^2 = \left(\frac{2\pi}{\ell}\right)^{2m} \sum_{k=-\infty}^{+\infty} k^{2m} |\hat{u}_k|^2\,.$$

Let $\hat{a}_k^2 = \frac{1}{T}\int_0^T |\hat{u}_k(t)|^2 dt$; then we have

$$\frac{1}{T}\int_0^T \|u\|_m^2 dt = \left(\frac{2\pi}{\ell}\right)^{2m} \sum_{k=-\infty}^{+\infty} k^{2m} \hat{a}_k^2\,.$$

Combining this formula with the results of sections 2 and 3 we see that for any $m \geqslant 1$ and for any δ small enough $\big($namely $\delta < \delta_0 = \min(1, \delta_0', \delta_0'')\big)$, it follows that

$$\tilde{c}_m \delta^{-2m+1} < \sum_{k=-\infty}^{+\infty} k^{2m} \hat{a}_k^2 < \widetilde{C}_m \delta^{-2m+1}\,, \tag{$*$}$$

where $\tilde{c}_m = \left(\frac{\ell}{2\pi}\right)^{2m} c_m^2$ and $\widetilde{C}_m = \left(\frac{\ell}{2\pi}\right)^{2m} C_m^2$. We define the constant V to be $V = \frac{\ell}{4\pi^2} \left|\frac{du_0}{dx}\right|_{L_1}^2$. Then

$$|\hat{u}_k| = \left| \frac{i\sqrt{\ell}}{2\pi k} \int \frac{\partial u}{\partial x} \exp\left(\frac{-2\pi i k x}{\ell}\right) dx \right| \leqslant \frac{\sqrt{\ell}}{2\pi |k|} \left|\frac{\partial u}{\partial x}\right|_{L_1} \leqslant \frac{V^{1/2}}{|k|}\,.$$

In the last estimate we have used inequality (A.4). Thus we have

$$\hat{a}_k^2 \leqslant \frac{V}{k^2}\,.$$

Let s be a nonnegative real and θ be a positive real. By $\hat{A}_{s,\theta}^2$ we denote the average of the values $\hat{a}_k^2$ over $|k| \in (\delta^{-s+\theta}; \delta^{-s-\theta}]$. Suppose $\delta^\theta < \frac{1}{2}$. This condition ensures that set of indexes k, over which the averaging takes place is not empty. Thus, using the simple fact that $\frac{1}{k^2} < \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} \frac{dx}{x^2}$, we have

$$\hat{A}_{s,\theta}^2 < \frac{V \int_{[\delta^{-s+\theta}]+\frac{1}{2}}^{[\delta^{-s-\theta}]+\frac{1}{2}} \frac{1}{x^2} dx}{[\delta^{-s-\theta}] - [\delta^{-s+\theta}]} = \frac{V}{\left([\delta^{-s-\theta}] + \frac{1}{2}\right) - \left([\delta^{-s+\theta}] + \frac{1}{2}\right)} < 4V\delta^{2s}$$

and estimate (1.6) is proved.

Let us turn to estimate (1.7). Suppose $1 < s - \theta < s < s + \theta$, $\delta^\theta < \frac{1}{2}$, and $(*)$ holds. Then

$$\sum_{|k| > \delta^{-s+\theta}} \hat{a}_k^2 \leqslant (\delta^{s-\theta})^{2m} \sum_{|k| > \delta^{-s+\theta}} k^{2m} \hat{a}_k^2 \leqslant (\delta^{s-\theta})^{2m} \sum_{k=-\infty}^{+\infty} k^{2m} \hat{a}_k^2 \leqslant \widetilde{C}_m \delta^{2m(s-\theta-1)+1}.$$

Thus we have

$$\hat{A}^2_{s,\theta} \leqslant \frac{\sum_{|k|>\delta^{-s+\theta}} \hat{a}_k^2}{2\left([\delta^{-s-\theta}]-[\delta^{-s+\theta}]\right)} \leqslant \frac{\widetilde{C}_m \delta^{2m(s-\theta-1)+1}}{\delta^{-s-\theta}} < \widetilde{C}_m \delta^{2+2m(s-\theta-1)}$$

and estimate (1.7) is proved.

Now let us turn to (1.8). Suppose $\frac{2V+\widetilde{C}_2}{\tilde{c}_1}\delta^\theta < \frac{1}{2}$ and $(*)$ holds. We have

$$\sum_{|k|>\delta^{-1-\frac{\theta}{2}}} k^2\hat{a}_k^2 \leqslant \delta^{2+\theta} \sum_{|k|>\delta^{-1-\frac{\theta}{2}}} k^4\hat{a}_k^2 \leqslant \widetilde{C}_2\delta^{-1+\theta}$$

and

$$\sum_{|k|\leqslant\delta^{-1+\theta}} k^2\hat{a}_k^2 \leqslant 2 \sum_{0<k\leqslant\delta^{-1+\theta}} V \leqslant 2V\delta^{-1+\theta} \ .$$

Therefore we obtain

$$\sum_{\delta^{-1+\theta}<|k|\leqslant\delta^{-1-\theta}} \hat{a}_k^2 \geqslant \delta^{2+\theta} \sum_{\delta^{-1+\theta}<|k|\leqslant\delta^{-1-\frac{\theta}{2}}} k^2\hat{a}_k^2 =$$

$$= \delta^{2+\theta}\left(\sum_{k=-\infty}^{+\infty} k^2\hat{a}_k^2 - \sum_{|k|\leqslant\delta^{-1+\theta}} k^2\hat{a}_k^2 - \sum_{|k|>\delta^{-1-\frac{\theta}{2}}} k^2\hat{a}_k^2\right) \geqslant$$

$$\geqslant \tilde{c}_1\delta^{1+\theta}\left(1-\tfrac{2V+\widetilde{C}_2}{\tilde{c}_1}\delta^\theta\right) \geqslant \frac{\tilde{c}_1}{2}\delta^{1+\theta} \ .$$

Thus we have $\hat{A}^2_{1,\theta} \geqslant \frac{\frac{1}{2}\tilde{c}_1\delta^{1+\theta}}{2\delta^{-1-\theta}} = \frac{\tilde{c}_1}{4}\delta^{2+2\theta}$ and estimate (1.8) is proved.

Appendix

Let the function $u = u^\delta(t,x)$ satisfy the equation

$$\frac{\partial}{\partial t}u + \frac{\partial f(u)}{\partial x} = \delta u_{xx} \tag{A.1}$$

in the cylinder $[0,+\infty)\times S$, with the initial state

$$u^\delta(0,x) = u_0(x) \in C^\infty(S) \ . \tag{A.2}$$

Here $f:\mathbb{R}\to\mathbb{R}$ is an arbitrary C^∞-smooth function (not necessarily convex), $\delta>0$. Hence (see [1]) the function u^δ remains C^∞-smooth for all $t>0$.

We recall that if a function $v(t,x)$ satisfies the equation

$$v_t + \psi(t,x)v_x = \delta v_{xx} \ ,$$

where $\psi(t,x)$ is C^∞-smooth then a maximum principle holds, i.e., for any $t_1<t_2$ we have

$$\inf v(t_1,\cdot) \leqslant \inf v(t_2,\cdot) \leqslant \sup v(t_2,\cdot) \leqslant \sup v(t_1,\cdot) \ .$$

Let the symbol $\dot{u}$ denote either the function $\frac{\partial u}{\partial t}$ or the function $\frac{\partial u}{\partial x}$ (the following lemma holds for both these cases). We consider the L_1-norm of the function $\dot{u}$ over the circle S as a function of t, $|\dot{u}|_{L_1}(t) = \int|\dot{u}(t,x)|dx$.

Lemma A. *The function* $t \mapsto |\dot u|_{L_1}(t)$ *is non-increasing.*
Proof. The function $\dot u(t,x)$ satisfies the equation

$$\dot u_t + (f'(u)\dot u)_x = \delta \dot u_{xx} \ .$$

We multiply this by a function $\varphi(t,x) \in [T_1,T_2]\times S$, where $T_1 < T_2$ and integrate by part:

$$\int \dot u\varphi\Big|_{t=T_2} dx = \int \dot u\varphi\Big|_{t=T_1} dx + \int_{T_1}^{T_2}\int \dot u(\varphi_t + f'(u)\varphi_x + \delta\varphi_{xx})dxdt \ . \qquad (A.3)$$

We consider the Cauchy problem for the equation

$$\varphi_t + f'(u)\varphi_x + \delta\varphi_{xx} = 0$$

in the cylinder $[T_1,T_2]\times S$. An initial state should be given at the *top* of the cylinder. We put

$$\varphi(T_2,x) = \mathrm{sign}^{\varepsilon}(\dot u(T_2,x)) \ ,$$

where the function $\mathrm{sign}^{\varepsilon}(\dot u(T_2,\cdot))$ is a smooth approximation of the function $\mathrm{sign}(\dot u(T_2,\cdot))$ such that $|\,\mathrm{sign}^{\varepsilon}(\dot u(T_2,x))| \leqslant 1 \quad \forall x$ and distance between these two function in the $L_1(S)$-norm less then ε. Now, by virtue of the maximum principle, we have that $|\varphi(t,x)| \leqslant 1$ for any $(t,x) \in [T_1,T_2]\times S$. Substituting $\varphi(t,x)$ into (A.3) we have

$$\int \dot u(T_2,x)\,\mathrm{sign}^{\varepsilon}(\dot u(T_2,x))dx = \int \dot u\varphi\Big|_{t=T_1} dx \leqslant \int |\dot u(T_1,x)|dx \ .$$

Letting ε tend to 0, we have

$$\int |\dot u|\Big|_{t=T_2} dx \leqslant \int |\dot u|\Big|_{t=T_1} dx \ . \qquad \square$$

It follows from this lemma that for any $t \geqslant 0$ and $\delta > 0$ we have

$$\int \left|\frac{\partial u^\delta}{\partial x}(t,x)\right| dx \leqslant \int \left|\frac{du_0}{dx}\right| dx \ . \qquad (A.4)$$

For $\delta \in (0,1)$ we have

$$\int \left|u^\delta(t_2,x) - u^\delta(t_1,x)\right| dx \leqslant C|t_2 - t_1| \ , \qquad (A.5)$$

where the constant C does not depend on δ. We can put

$$C = \int \left|\frac{df(u_0)}{dx}\right| dx + \int \left|\frac{d^2u_0}{dx^2}\right| dx \ .$$

We now investigate the convergence of the solution of the problem (A.1), (A.2) as $\delta \to 0$.
Theorem. *For any* $T \geqslant 0$ *and* $\delta \geqslant 0$ *we have*

$$\int |u^\delta(T,x) - u^0(T,x)|dx \leqslant \frac{4}{\sqrt 3}\sqrt{T\delta}\int \left|\frac{du_0}{dx}(x)\right| dx \ . \qquad (A.6)$$

In what follows we omit the index δ and we denote the limiting function u^0 by v.

Lemma 1. *For any real number a and for any C^2-smooth, nonnegative function $\psi(t,x)$ we have*

$$\int |u(T,x)-a|\psi(T,x)dx \leqslant \int |u(0,x)-a|\psi(0,x)dx+$$
$$+\int_0^T \int F(u(t,x),a)\psi_x(t,x)+|u(t,x)-a|(\psi_t(t,x)+\delta\psi_{xx}(t,x))dxdt\,, \quad \text{(A.7)}$$

where $F(u,a)=\mathrm{sign}(u-a)(f(u)-f(a))$.

Proof. We multiply equation (A.1) by a function $\Phi_n'(u)\psi(t,x)$, where $\Phi_n(\cdot)$ is C^2-smooth, convex ($\Phi_n''(\cdot)\geqslant 0$) functions. Then we integrate over the cylinder $[0,T]\times S$:

$$\int_0^T\int \Phi_n'(u)u_t\psi dxdt+\int_0^T\int \Phi_n'(u)f'(u)u_x\psi dxdt=\delta\int_0^T\int u_{xx}\Phi_n'(u)\psi dxdt\,.$$

Noting that $\Phi_n'(u)f'(u)u_x=\frac{\partial}{\partial x}\int_0^u\Phi_n'(\xi)f'(\xi)d\xi$ and integrating by parts, we have:

$$\int \Phi_n(u)\psi\Big|_{t=T}dx-\int \Phi_n(u)\psi\Big|_{t=0}dx-$$
$$-\int_0^T\int \Phi_n(u)\psi_t dxdt-\int_0^T\int\left(\int_0^u \Phi_n'(\xi)f'(\xi)d\xi\right)\psi_x dxdt=$$
$$=-\delta\int_0^T\int u_x^2\Phi_n''(u)\psi dxdt-\delta\int_0^T\int \Phi_n'(u)u_x\psi_x dxdt\,,$$

or

$$\int \Phi_n(u)\psi\Big|_{t=T}dx\leqslant\int \Phi_n(u)\psi\Big|_{t=0}dx+\int_0^T\int\left(\int_0^u \Phi_n'(\xi)f'(\xi)d\xi\right)\psi_x dxdt+$$
$$+\int_0^T\int \Phi_n(u)(\psi_t+\delta\psi_{xx})dxdt\,.$$

We now suppose that the $\Phi_n(\cdot)$ approximate the function $|\cdot-a|$ in the supremum norm, such that

$$\frac{d\Phi_n(u)}{du}\to\frac{d}{du}|u-a|=\begin{cases}1 & \text{if } u>a\\ -1 & \text{if } u<a\end{cases}$$

in $L_1(u_-,u_+)$, where $u_-=\min u_0$, $u_+=\max u_0$. Thus we can pass to the limit and obtain the inequality (A.7). □

It follows from the maximum principle for u^δ and inequality (A.4) that the family of the functions

$$\{u^\delta(t,\cdot)\}_{\delta\in(0,1);\,t\in[0,T]} \qquad (\star)$$

is uniformly bounded and has uniformly bounded variation. By Helly's selection principle (see [7]) any infinite subset of this family contains a pointwise convergent sequence. By Lebesgue's bounded convergence theorem this sequence converges in $L_1(S)$. This means that the family $(\star)$ is precompact in $L_1(S)$. From this, inequality (A.5) and Ascoli-Arzelà theorem it follows that the family of the functions $\{u^\delta(t,x)\}_{\delta\in(0,1)}$ is precompact in $C\big([0,T];L_1(S)\big)$. Thus there exists an element $v\in C\big([0,T];L_1(S)\big)$ and a sequence $\delta_k \xrightarrow[k\to\infty]{} 0$ such that $u^{\delta_k}\to v$ in $C\big([0,T];L_1(S)\big)$.

We note that this convergence and the maximum principle for u^{δ_k} imply that for any t we have:

$$u_- \leqslant \operatorname{ess\,inf} v(t,\cdot) \leqslant \operatorname{ess\,sup} v(t,\cdot) \leqslant u_+ \ ,$$

otherwise $|u^{\delta_k}(t,\cdot)-v(t,\cdot)|_{L_1}$ does not tend to 0. For the next step we need the inequalities

$$\big||u(t,x)-a|-|v(t,x)-a|\big| \leqslant |u(t,x)-v(t,x)|$$

and

$$\big|F(u(t,x),a)-F(v(t,x),a)\big| \leqslant \max_{[u_-,u_+]}|f'(\cdot)||u(t,x)-v(t,x)| \quad \text{ for a.a. } (t,x)\,.$$

(The first is just the triangle inequality. To prove the second, one should consider two cases: a) a is between u and v; b) u and v are either both greater than or both less than a. For each t the second inequality holds for almost all x.)

We pass to the limit in inequality (A.7) along the sequence $\delta_k\to 0$ (the function $\psi(t,x)$ and the initial state u_0 are fixed):

$$\int |v(T,\tilde x)-b|\tilde\psi(T,\tilde x)d\tilde x \leqslant \int |v(0,\tilde x)-b|\tilde\psi(0,\tilde x)d\tilde x + \\ + \int_0^T\int F(v(\tilde t,\tilde x),b)\tilde\psi_{\tilde x}(\tilde t,\tilde x)+|v(\tilde t,\tilde x)-b|\tilde\psi_{\tilde t}(\tilde t,\tilde x)d\tilde x d\tilde t\,. \quad \text{(A.8)}$$

Here we have replaced a with b and t,x,ψ with $\tilde t,\tilde x,\tilde\psi$ for purposes of the remainder of the proof.

Lemma 2. *Suppose that $\Omega : S\to\mathbb{R}$ is an arbitrary nonnegative C^2-smooth function on the circle S; then*

$$\int\!\!\int |u(T,x)-v(T,\tilde x)|\Omega_\varepsilon(x-\tilde x)dxd\tilde x \leqslant \int\!\!\int |u(0,x)-v(0,\tilde x)|\Omega_\varepsilon(x-\tilde x)d\tilde x dx + \\ +\delta\int_0^T\int\!\!\int |u(t,x)-v(t,\tilde x)|\Omega''_\varepsilon(x-\tilde x)dxd\tilde x dt\,.$$

Proof. Suppose $\omega_\varepsilon:\mathbb{R}\to\mathbb{R}$ is a C^2-smooth, even, non-negative function such that $\omega_\varepsilon(z)\equiv 0$ for $|z|\geqslant\varepsilon$ and

$$\int_{-\infty}^{+\infty}\omega_\varepsilon(z)dz = 1\,.$$

We note that for the function $\hat\psi(t,\tilde t,x,\tilde x)=\omega_\varepsilon(t-\tilde t)\Omega(x-\tilde x)$ we have $\frac{\partial}{\partial t}\hat\psi = -\frac{\partial}{\partial\tilde t}\hat\psi$ and $\frac{\partial}{\partial x}\hat\psi=-\frac{\partial}{\partial\tilde x}\hat\psi$. We substitute $a=v(\tilde t,\tilde x)$, $\psi(t,x)=\omega_\varepsilon(t-\tilde t)\Omega(x-\tilde x)$ into (A.7) and integrate with respect to $(\tilde t,\tilde x)$ over $[0,T]\times S$. Also, we substitute $b=u(t,x)$, $\tilde\psi(\tilde t,\tilde x)=\omega_\varepsilon(\tilde t-t)\Omega(\tilde x-x)$ into inequality (A.8) and integrate with respect to (t,x) over $[0,T]\times S$. Then we add the resulting two inequalities.

$$
\begin{gathered}
\int_0^T\int\int |u(T,x)-v(\tilde t,\tilde x)|\omega_\varepsilon(T-\tilde t)\Omega(x-\tilde x)\,dx\,d\tilde x\,d\tilde t+\\
+\int_0^T\int\int |v(T,\tilde x)-u(t,x)|\omega_\varepsilon(T-t)\Omega(x-\tilde x)\,d\tilde x\,dx\,dt \leqslant\\
\leqslant \int_0^T\int\int |u(0,x)-v(\tilde t,\tilde x)|\omega_\varepsilon(\tilde t)\Omega(x-\tilde x)\,dx\,d\tilde x\,d\tilde t+\\
+\int_0^T\int\int |v(0,\tilde x)-u(t,x)|\omega_\varepsilon(t)\Omega(x-\tilde x)\,d\tilde x\,dx\,dt+\\
+\delta\int_0^T\int_0^T\int\int |u(t,x)-v(\tilde t,\tilde x)|\omega_\varepsilon(t-\tilde t)\tfrac{\partial^2}{\partial x^2}\Omega(x-\tilde x)\,dx\,d\tilde x\,dt\,d\tilde t\ .
\end{gathered}
$$

Letting ε tend to 0, we get the statement of the lemma. $\square$

Lemma 3. Suppose $\Psi:\mathbb{R}\to\mathbb{R}$ is a nonnegative C^2-smooth function with bounded support; then

$$
\int_{-\infty}^{\infty}\int |u(T,x+y)-v(T,x)|\Psi(y)dxdy \leqslant \int_{-\infty}^{\infty}\int |u(0,x+y)-v(0,x)|\Psi(y)dxdy+
$$
$$
+\,\delta\int_0^T\int_{-\infty}^{\infty}\int |u(t,x+y)-v(t,x)|\Psi''(y)dxdydt\ . \quad \text{(A.9)}
$$

Proof. We apply the previous lemma with $\Omega(\cdot)=\sum_{k=-\infty}^{+\infty}\Psi(\cdot+k\,\ell)$ replacing $x-\tilde x$ with y and $\tilde x$ with x. $\square$

To get the desired result from (A.9) we apply the triangle inequality in the following form:

$$
|u(t,x+y)-v(t,x)| = |u(t,x)-v(t,x)| + w(t,x,y)\ ,
$$

where $|w(t,x,y)|\leqslant |u(t,x+y)-u(t,x)|$. Using equality $u(0,x)\equiv v(0,x)$ and $\int_{-\infty}^{\infty}\Psi''(y)dy=0$, we have

$$
\int |u(T,x)-v(T,x)|dx\int_{-\infty}^{+\infty}\Psi(y)dy \leqslant \int_{-\infty}^{+\infty}\int |u(T,x)-u(T,x+y)|dx\Psi(y)dy+
$$
$$
+\int_{-\infty}^{+\infty}\int |u(0,x)-u(0,x+y)|dx\Psi(y)dy+\delta\int_0^T\int_{-\infty}^{+\infty}\int |u(t,x)-u(t,x+y)|dx|\Psi''(y)|dydt. \tag{A.10}
$$

Further we shall assume that $\int_{-\infty}^{+\infty}\Psi(y)dy=1$. Using inequalities

$$
\int |u(t,x)-u(t,x+y)|dx = \int\left|\int_0^y \frac{\partial u}{\partial x}(t,x+\xi)d\xi\right|dx \leqslant
$$
$$
\leqslant \left|\int_0^y\int\left|\frac{\partial u}{\partial x}(t,x+\xi)\right|dxd\xi\right| = |y|\int\left|\frac{\partial u}{\partial x}(t,x)\right|dx \overset{(A.4)}{\leqslant} |y|\int\left|\frac{du_0}{dx}\right|dx\ ,
$$

we rewrite (A.10) as follows:

$$\int |u(T,x) - v(T,x)|dx \leqslant \left(2 \int\limits_{-\infty}^{+\infty} |y|\Psi(y)dy + T\delta \int\limits_{-\infty}^{+\infty} |y||\Psi''(y)|dy \right) \int \left| \frac{du_0}{dx} \right| dx \ . \tag{A.11}$$

Now let the functions $\Psi(y)$ approximate the function $\max\Big(\frac{1}{\varepsilon}\big(1 - \frac{|y|}{\varepsilon}\big), 0\Big)$. Then the expression in parentheses in (A.11) tends to

$$2\int_{-\varepsilon}^{\varepsilon} |y|\tfrac{1}{\varepsilon}\big(1 - \tfrac{|y|}{\varepsilon}\big)dy + 2\delta T\tfrac{1}{\varepsilon} = \tfrac{2}{3}\left(\varepsilon + \tfrac{3T\delta}{\varepsilon}\right) \ .$$

Minimising with respect to ε, we get estimate (A.6). The theorem is proved. □

References

[1] Ladyzhenskaya, O.A.; Solonnikov, V.A.; Ural'ceva, N.N. *Linear and quasi-linear equations of parabolic type.* Nauka, M., 1967. English translation: Providence, (R.I.): Amer. Math. Soc. 1968. (Transl. Math. Monogr., vol. **23**).

[2] Kruzhkov S. N. *First order quasilinear equations in several independent variables.* Mat. Sbornik (N.S.) **81**(**123**) no 2, 228-255 (1970). English translation: Math. USSR Sb.**10**, 217-243 (1970).

[3] Kuznetsov N. N. *Accuracy of some approximate methods for computing the weak solutions of a first-order quasilinear equation.* Zh. Vych. Mat. i Mat. Phys. **16**, no 6, 1489-1502 (1976). English translation: USSR Comp. Math. and Math. Phys. **16**, no 6, 105-119 (1976).

[4] Kolmogorov A. N. *On inequalities for supremums of successive derivatives of a function on an infinite interval.* Paper 40 in "Selected works of A.N. Kolmogorov, vol.1" Moscow, Nauka 1985. Engl. translation Kluwer, 1991.

[5] Hörmarder L. *Lectures on Nonlinear Hyperbolic Differential Equations.* Mathématiques & Applications, Vol. **26**. Springer. 1997.

[6] Lax P.D. *Hyperbolic systems of conservation laws II.* Comm. Pure Appl. Math., **10**, 537-566 (1957).

[7] Kolmogorov A. N., Fomin S. V. *Introductory real analysis.* Dover Publ., New York. 1975.

[8] Frish U. *Turbulence. The legacy of A.N. Kolmogorov.* Cambridge Univ. Press, 1995.

[9] Kuksin S. *Spectral Properties of Solutions for Nonlinear PDE's in the Turbulent Regime.* Geometric and Functional Analysis, Vol. **9**, No. 1, 141-184 (1999).

[10] E W., Khanin K., Mazel A., Sinai Ya. *Invariant measures for Burgers equation with stochastic forcing.* "Annals of Mathematics" Vol. **151**., part 3., 877-960 (2000).