On Multidimensional Burgers Type Equations with Small Viscosity

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Abstract. We consider the Cauchy problem for a multidimensional Burgers type equation with periodic boundary conditions. We obtain upper and lower bounds for derivatives of solutions for this equation in terms of powers of the viscosity and discuss how these estimates relate to the Kolmogorov–Obukhov spectral law. Next we use the estimates obtained to get certain bounds for derivatives of solutions of the Navier-Stokes system.

Keywords: Kolmogorov–Obukhov spectral law, bounds for derivatives, degenerate state.

1 Introduction

We study the dynamics of m-dimensional vector field $\boldsymbol{u} = \boldsymbol{u}(t, \boldsymbol{x})$ on the n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/_{(\ell\mathbb{Z})^n}$ described by the equation

$$\partial_t \boldsymbol{u} + \nabla_{\boldsymbol{f}(\boldsymbol{u})} \boldsymbol{u} = \nu \Delta \boldsymbol{u} + \boldsymbol{h}(t, \boldsymbol{x})\,. \tag{1.1}$$

Here ν is a positive parameter ("the viscosity"), $\boldsymbol{f} : \mathbb{R}^m \to \mathbb{R}^n$ is a smooth map, $\boldsymbol{h}$ is a smooth forcing term and $\nabla_{\boldsymbol{f}(\boldsymbol{u})}$ is the derivative along the vector $\boldsymbol{f}(\boldsymbol{u})$, i.e., $\nabla_{\boldsymbol{f}(\boldsymbol{u})} \boldsymbol{u} = (\boldsymbol{f}(\boldsymbol{u}) \cdot \nabla)\boldsymbol{u}$.

If $m = n$ and $\boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{u}$, we have the usual forced Burgers equation. In a potential case (i.e., if the initial state $\boldsymbol{u}_0(\boldsymbol{x}) = \boldsymbol{u}(0, \boldsymbol{x})$ and the field $\boldsymbol{h}$ are gradients of some functions) this equation can be reduced to a linear parabolic equation.

As it is shown in [1], [12], appropriate bounds for derivatives imply estimates for averaged spectral characteristics of the flow. The purpose of this work is to obtain such bounds for solutions of the Cauchy problem for the generalised m-n multidimensional Burgers equation (1.1).

We describe notations used in this article. If $\boldsymbol{v}$ is a vector in $\mathbb{R}^s$, $\mathbb{Z}^s$ or $\mathbb{C}^s$, then $|\boldsymbol{v}|$ denotes its Euclidean (Hermitian) norm $|\boldsymbol{v}|^2 = \sum_{i=1}^s |v_i|^2$. If we have to stress the dimension, we denote the norm in $\mathbb{R}^s$ as $|\boldsymbol{v}|_{\mathbb{R}^s}$, etc. By $B(r)$ we denote the ball of radius r centered at the origin. If $A : \mathbb{R}^{s_1} \to \mathbb{R}^{s_2}$ is a linear map then $\|A\|$ denotes the operator-norm of this map associated with the Euclidean norms $|\cdot|$ on $\mathbb{R}^{s_1}$ and $\mathbb{R}^{s_2}$. If $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x})$, then we write

$$|\boldsymbol{v}| = \sup_{\boldsymbol{x}} |\boldsymbol{v}(\boldsymbol{x})| = \sup_{\boldsymbol{x}} \Big(\sum |v_i(\boldsymbol{x})|^2\Big)^{1/2}. \tag{1.2}$$

Sometimes we will denote this norm by $|\cdot|_{L_\infty}$. If $\boldsymbol{v} = \boldsymbol{v}(t, \boldsymbol{x})$, then $|\boldsymbol{v}| = \sup_{\boldsymbol{x}} |\boldsymbol{v}(t, \boldsymbol{x})|$ is a function of t. For a multi-index α we denote $|\alpha| = \sum |\alpha_i|$.

We set

$$H(t) = \int_0^t \sup_{\boldsymbol{x} \in \mathbb{T}^n} |\boldsymbol{h}(\tau, \boldsymbol{x})|_{\mathbb{R}^m}\, d\tau\,. \tag{1.3}$$

We also denote

$$[\boldsymbol{f}]_{C^k(r)} = \max_{|\beta|=k} \sup_{\{\boldsymbol{u} \in \mathbb{R}^m : |\boldsymbol{u}| \leqslant r\}} \Big(\sum_{j=1}^n \Big|\tfrac{\partial^k}{\partial \boldsymbol{u}^\beta} f_j\Big|^2\Big)^{1/2}, \tag{1.4}$$

and

$$\|\boldsymbol{u}\|_k^2 = \int_{\mathbb{T}^n} \sum_{i=1}^m u_i (-\Delta)^k u_i d\boldsymbol{x} = \sum_{i=1}^m \sum_{|\alpha|=k} \tbinom{|\alpha|}{\alpha} \big|D^\alpha u_i\big|^2_{L_2(\mathbb{T}^n)} = \sum_{i=1}^m \sum_{j_1,\dots,j_k=1}^n \big|\tfrac{\partial^k u_i}{\partial x_{j_1} \cdots \partial x_{j_k}}\big|^2_{L_2(\mathbb{T}^n)}. \tag{1.5}$$

Here $k \geqslant 0$ is an integer and $\binom{|\alpha|}{\alpha} = \binom{|\alpha|}{\alpha_1,\dots,\alpha_n} = \frac{(\alpha_1+\dots+\alpha_n)!}{\alpha_1! \cdots \alpha_n!}$ are coefficients in the generalised binomial expansion $(x_1 + x_2 + \ldots + x_n)^k = \sum_{|\alpha|=k} \binom{k}{\alpha} \boldsymbol{x}^\alpha$. If $\boldsymbol{u} = \boldsymbol{u}(t, \boldsymbol{x})$, then $\|\boldsymbol{u}\|_k = \|\boldsymbol{u}(t, \cdot)\|_k$ is a function of t.

Our main results are stated in the following two theorems, where $\boldsymbol{u}(t, \boldsymbol{x})$, $t \geqslant 0$, is any smooth solution for the equation (1.1):

Theorem 1. *For any $k \geqslant 0$, $t \geqslant 0$, and $\nu > 0$ we have*

$$\|\boldsymbol{u}(t,\cdot)\|_k \leqslant R_k(t) \max\left\{\frac{1}{\nu^k}, \frac{\|\boldsymbol{u}_0\|_k}{R_k(0)}, \frac{\sup_{[0,t]}\|\boldsymbol{h}\|_{k-1}}{R_k(0)}\right\}. \tag{1.6}$$

Here $\|\boldsymbol{h}\|_{-1} = 0$, $R_0(t) = (|\boldsymbol{u}_0| + H(t))\ell^{n/2}$ *and*

$$R_k(t) = \left(1 + C_{k,m,n} \max_{s=0\ldots k-1}\{[\boldsymbol{f}]_{C^s(|\boldsymbol{u}_0|+H(t))}\,(|\boldsymbol{u}_0| + H(t))^s\}\right)^k (|\boldsymbol{u}_0| + H(t))\ell^{n/2},$$

where the constant $C_{k,m,n}$ depends on k, m, n only.

Definition 1. *The vector field $\boldsymbol{u}_0$ is degenerate with respect to equation (1.1) if the matrix $\frac{\partial \boldsymbol{f}(\boldsymbol{u}_0)}{\partial \boldsymbol{x}}$ (this is an $n \times n$ matrix, which depends on $\boldsymbol{x}$) is everywhere nilpotent, i.e., for each point $\boldsymbol{x}$ some power of this matrix is equal to 0.*

Theorem 2. *Suppose that the initial state $\boldsymbol{u}_0$ is a non-degenerate vector field. Then there exist ν-independent positive real constants T, c and r_2, r_3, r_4, ... such that:*

If $H(T) < \frac{c}{2}$ then $\forall \nu > 0$ and $\forall k \geqslant 2$, we have:

$$\max_{j=1\ldots n} \frac{1}{T}\int_0^T \sup_{\boldsymbol{x}\in\mathbb{T}^n} \left|\frac{\partial^k \boldsymbol{u}}{\partial {x_j}^k}(t,\boldsymbol{x})\right|_{\mathbb{R}^m} dt \geqslant \frac{r_k}{\nu^{k/2}}. \tag{1.7}$$

The constants T, c, r_2, r_3,... depend on the non-degeneracy of the initial state $\boldsymbol{u}_0$ in quite complicated way (see (3.11), (3.16), (3.19), and (3.33)). The nearer is $\boldsymbol{u}_0$ to set of degenerate vector functions, the bigger is T and the smaller are c, r_2, r_3,.... If $\boldsymbol{u_0} = \lambda \boldsymbol{v_0}$ and $\lambda \to 0$, then $T \propto \lambda^{-1}$, $r_k \propto \lambda^{k/2}$, c does not depend on λ.

In section 3 we give an example of a degenerate non-constant initial state for which derivatives of the solution are bounded by ν-independent constants for all $t \geqslant 0$. Moreover, for the two dimensional case ($m = n = 2$) and for $\boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{u}$, $\boldsymbol{h} \equiv 0$ we show that any solution with a degenerate initial state retains bounded derivatives. This fact is based on a result from the classical geometry due to Pogorelov – Hartman – Nirenberg, known as the "Cylinder Theorem". In section 3 we show that in the case $m = n$ and $\boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{u}$, any non-constant *potential* initial state is non-degenerate.

The exponents of viscosity ν in inequalities (1.6) and (1.7) are not sharp. In the one-dimensional case ($m = n = 1$) sharp values for the exponents can be obtained. Namely, it is shown in [1] that for $k \geqslant 1$ we have

$$\|\boldsymbol{u}\|_k \leqslant C_k\left(\tfrac{1}{\nu}\right)^{k-1/2}, \quad \left(\frac{1}{T}\int_0^T \|\boldsymbol{u}\|_k^2 dt\right)^{1/2} \geqslant c_k\left(\tfrac{1}{\nu}\right)^{k-1/2}.$$

As a consequence of these inequalities one can get bounds for magnitudes of the derivatives:

$$\left|\frac{d^k u}{dx^k}\right|_{L_\infty} \leqslant \frac{C'_k}{\nu^k}, \quad \frac{1}{T}\int_0^T \left|\frac{d^k u}{dx^k}\right|_{L_\infty} dt \geqslant \frac{c'_k}{\nu^k}.$$

The first inequality for $k = 0$ follows by the maximum principle and for $k \geqslant 1$ – by the inequality $|v|_{L_\infty} \leqslant |v|_{L_2}^{1/2}|v_x|_{L_2}^{1/2}$ which holds for any periodic function v with zero meanvalue, see [1], sect. 3. To derive the second inequality (see [1] and formula (3.7) there) we use the well-known fact that for periodic solutions of 1D Burgers-type equations the quantity $\left|\frac{du}{dx}\right|_{L_1}$ is bounded uniformly in t (see e.g. the appendix in [1]). Then for $k = 1$ the second estimate follows by the Hölder inequality $|u_x|_{L_2}^2 \leqslant |u_x|_{L_\infty}|u_x|_{L_1}$, while for $k > 1$ it follows by interpolation with the upper bound for $k = 0$.

This article is organised as follows. In section 2 we prove the upper estimates (1.6) (theorem 1). Section 3 is devoted to proving the lower bounds (1.7) (theorem 2). In section 4 we obtain some results on behaviour of Fourier coefficients of solutions that can be extracted from the bounds (1.6) and (1.7). Assuming that there is a Kolmogorov–Obukhov type spectral asymptotics for the Fourier coefficients of solutions of (1.1), we get bounds for the exponents of the spectral law and for the Kolmogorov dissipation scale. In section 5, treating the Navier-Stokes system as a partial case of (1.1), we derive lower bounds for derivatives of its solutions.

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2 Upper estimates

In this section we prove theorem 1. The componentwise representation of (1.1) is

$$\frac{\partial}{\partial t}u_i + \sum_{j=1}^{n} f_j(u_1,\dots,u_m)\frac{\partial u_i}{\partial x_j} = \nu\Delta u_i + h_i(t,x_1,\dots,x_n)\,, \tag{2.1}$$

where $i = 1,\dots,m$.

Lemma 1. *Let T and ν be any positive numbers. Let $v = v(t,\boldsymbol{x})$ be a continuous function on $[0,T]\times\mathbb{T}^n$ with the continuous derivatives v_t, v_{x_j} and $v_{x_jx_j}$ for any $j=1,\dots,n$. Let $V_j = V_j(t,\boldsymbol{x})$, $j=1,\dots,n$ and $g = g(t,\boldsymbol{x})$ be continuous functions on $[0,T]\times\mathbb{T}^n$. Suppose that on $[0,T]\times\mathbb{T}^n$ we have the following partial differential inequality:*

$$v_t + \sum_{j=1}^{n} V_j\frac{\partial v}{\partial x_j} \leqslant \nu\Delta v + g(t,x_1,\dots,x_n).$$

Then for any $(t,\boldsymbol{x})\in[0,T]\times\mathbb{T}^n$ we have

$$v(t,\boldsymbol{x}) \leqslant \max_{\boldsymbol{y}\in\mathbb{T}^n} v(0,\boldsymbol{y}) + \int_0^t \max_{\boldsymbol{y}\in\mathbb{T}^n} g(\tau,\boldsymbol{y})d\tau.$$

Proof. Making the substitution $v(t,\boldsymbol{x}) = \tilde{v}(t,\boldsymbol{x}) + q(t)$, where $q{:}[0,T]\to\mathbb{R}$ is a function such that $q'(t) = \max_{\boldsymbol{y}\in\mathbb{T}^n} g(t,\boldsymbol{y})$, we reduce this lemma to the case $g\equiv 0$. Now the statement of the lemma becomes a classic maximum principle, see e.g. [4]. □

Applying this lemma for $v(t,\boldsymbol{x}) = \sum a_i u_i(t,\boldsymbol{x})$ and $g(t,\boldsymbol{x}) = \sum a_i h_i(t,\boldsymbol{x})$ with appropriate unit vector $\boldsymbol{a}\in\mathbb{R}^m$ we obtain

$$|\boldsymbol{u}(t,\cdot)| \leqslant |\boldsymbol{u}_0| + H(t)\,. \tag{2.2}$$

Here the norm $|\cdot|$ is defined by (1.2) and $H(t)$ is defined by (1.3).

Since $\|\boldsymbol{u}\|_0 \leqslant \ell^{n/2}\,|\boldsymbol{u}|$, we have

$$\|\boldsymbol{u}(t,\cdot)\|_0 \leqslant \ell^{n/2}(|\boldsymbol{u}_0| + H(t))\,. \tag{2.3}$$

This proves (1.6) for $k=0$. Next, we multiply (2.1) by $(-\Delta)^k u_i$, take the sum over $i=1,\dots,m$, and integrate over the period (over the torus):

$$\tfrac{1}{2}\tfrac{d}{dt}\|\boldsymbol{u}\|_k^2 - b(\boldsymbol{f},\boldsymbol{u},(-\Delta)^k\boldsymbol{u}) = -\nu\|\boldsymbol{u}\|_{k+1}^2 + \Upsilon_2^k\,.$$

Here we denote

$$b(\boldsymbol{f},\boldsymbol{u},\boldsymbol{v}) = -\int_{\mathbb{T}^n}\sum_{\substack{j=1\dots n\\ i=1\dots m}} f_j\frac{\partial u_i}{\partial x_j}v_i d\boldsymbol{x}\,. \tag{2.4}$$

and

$$\Upsilon_2^k = \int_{\mathbb{T}^n}\sum_{i=1\dots m} h_i(-\Delta)^k u_i d\boldsymbol{x}\,.$$

Lemma 2. *For the functional b introduced above we have*

$$b(\boldsymbol{f}(\boldsymbol{u}),\boldsymbol{u},\boldsymbol{u}) \leqslant [\boldsymbol{f}]_{C^0(|\boldsymbol{u}|)}\,\|\boldsymbol{u}\|_1\,\|\boldsymbol{u}\|_0\,, \tag{2.5}$$

$$b(\boldsymbol{f}(\boldsymbol{u}),\boldsymbol{u},(-\Delta)\boldsymbol{u}) \leqslant [\boldsymbol{f}]_{C^0(|\boldsymbol{u}|)}\,\|\boldsymbol{u}\|_1\,\|\boldsymbol{u}\|_2\,, \tag{2.6}$$

and for any $k\geqslant 2$ we have

$$b(\boldsymbol{f}(\boldsymbol{u}),\boldsymbol{u},(-\Delta)^k\boldsymbol{u}) \leqslant C_{k,m,n}\max_{s=0,\dots,k-1}\{[\boldsymbol{f}]_{C^s(|\boldsymbol{u}|)}\,|\boldsymbol{u}|_{L_\infty}^s\}\,\|\boldsymbol{u}\|_k\,\|\boldsymbol{u}\|_{k+1}\,. \tag{2.7}$$

Proof. First we prove a general inequality on $b(\cdot,\cdot,\cdot)$:

$$|b(\boldsymbol{f},\boldsymbol{u},\boldsymbol{v})| \leqslant |\boldsymbol{f}|\,\|\boldsymbol{u}\|_1\,\|\boldsymbol{v}\|_0\,, \tag{2.8}$$

where $|\boldsymbol{f}| = \sup\big(\sum_{j=1}^n f_j^2\big)^{1/2}$ and the norms $\|\cdot\|_s$ are defined in (1.5). Up to a constant factor this inequality is obvious. Below we show that for the chosen norm the constant is equal to 1. By the definition (2.4) of $b(\cdot,\cdot,\cdot)$ and the Cauchy-Schwartz inequality we have

$$|b(\boldsymbol{f},\boldsymbol{u},\boldsymbol{v})| \leqslant \int\Big(\sum_{j=1}^n f_j^2\Big)^{1/2}\Big(\sum_{j=1}^n\Big(\sum_{i=1}^m \tfrac{\partial u_i}{\partial x_j}v_i\Big)^2\Big)^{1/2}dx \leqslant$$

now we again use the Cauchy-Schwartz inequality $(\sum a_i b_i)^2 \leqslant (\sum a_i^2)(\sum b_i^2)$ to continue as follows:

$$\leqslant |\boldsymbol{f}|\int\Big(\sum_{j=1}^n\Big(\sum_{i=1}^m\big(\tfrac{\partial u_i}{\partial x_j}\big)^2\Big)\Big(\sum_{i=1}^m v_i^2\Big)\Big)^{1/2}dx = |\boldsymbol{f}|\int\Big(\sum_{j=1}^n\sum_{i=1}^m\big(\tfrac{\partial u_i}{\partial x_j}\big)^2\Big)^{1/2}\Big(\sum_{i=1}^m v_i^2\Big)^{1/2}dx \leqslant$$

$$\leqslant |\boldsymbol{f}|\Big(\int\sum_{j=1}^n\sum_{i=1}^m\big(\tfrac{\partial u_i}{\partial x_j}\big)^2 d\boldsymbol{x}\Big)^{1/2}\Big(\int\sum_{i=1}^m v_i^2\,dx\Big)^{1/2} = |\boldsymbol{f}|\,\|\boldsymbol{u}\|_1\,\|\boldsymbol{v}\|_0\,.$$

The inequality (2.8) and therefore (2.5) are proved. Using $\|\Delta\boldsymbol{u}\|_0 = \|\boldsymbol{u}\|_2$ we arrive at (2.6).

Consider the case $k \geqslant 2$. By (2.4) we have

$$b(\boldsymbol{f},\boldsymbol{u},(-\Delta)^k\boldsymbol{u}) = (-1)^{k-1}\int\sum_{j_0,\ldots,j_k=1}^n\sum_{i=1}^m f_{j_0}(u_1,\ldots,u_m)\frac{\partial u_i}{\partial x_{j_0}}\frac{\partial^2}{\partial x_{j_1}^2}\cdots\frac{\partial^2}{\partial x_{j_k}^2}u_i\,d\boldsymbol{x}.$$

Integrating by parts $k-1$ times we obtain

$$b(\boldsymbol{f},\boldsymbol{u},(-\Delta)^k\boldsymbol{u}) = \int\sum_{j_0,\ldots,j_k=1}^n\sum_{i=1}^m\frac{\partial}{\partial x_{j_1}}\cdots\frac{\partial}{\partial x_{j_{k-1}}}\Big(f_{j_0}(u_1,\ldots,u_m)\frac{\partial u_i}{\partial x_{j_0}}\Big)\frac{\partial}{\partial x_{j_1}}\cdots\frac{\partial}{\partial x_{j_{k-1}}}\frac{\partial^2}{\partial x_{j_k}^2}u_i\,d\boldsymbol{x}.$$

Using the identity $\|\boldsymbol{u}\|_{k+1}^2 = \int\sum_{j_1,\ldots,j_{k-1}=1,\ldots,n}\sum_{i=1}^m\big(\sum_{j_k=1}^n\frac{\partial}{\partial x_{j_1}}\cdots\frac{\partial}{\partial x_{j_{k-1}}}\frac{\partial^2}{\partial x_{j_k}^2}u_i\big)^2 d\boldsymbol{x}$ we get

$$|b(\boldsymbol{f},\boldsymbol{u},(-\Delta)^k\boldsymbol{u})| \leqslant \Big(\int\sum_{j_0,\ldots,j_{k-1}=1}^n\sum_{i=1}^m\Big(\frac{\partial}{\partial x_{j_1}}\cdots\frac{\partial}{\partial x_{j_{k-1}}}\big(f_{j_0}(u_1,\ldots,u_m)\tfrac{\partial u_i}{\partial x_{j_0}}\big)\Big)^2 d\boldsymbol{x}\Big)^{1/2}\|\boldsymbol{u}\|_{k+1}\,.$$

Now to prove the lemma it suffices to verify the inequality

$$\int\Big|\frac{\partial}{\partial x_{j_1}}\cdots\frac{\partial}{\partial x_{j_{k-1}}}\big(f_{j_0}(u_1,\ldots,u_m)\tfrac{\partial u_i}{\partial x_{j_0}}\big)\Big|dx \leqslant C'_{k,m,n}\max_{s=0,\ldots,k-1}\{[\boldsymbol{f}]_{C^s(|\boldsymbol{u}|)}\,|\boldsymbol{u}|_{L_\infty}^s\}\,\|\boldsymbol{u}\|_k\,. \tag{2.9}$$

(Indeed, (2.9) implies (2.7) with $C_{k,m,n} = (mn^{k-1})^{1/2}C'_{k,m,n}$.) Expanding the brackets in (2.9) we get no more than $(m+1)(m+2)\cdots(m+k-1)$ terms of the form

$$\int_{\mathbb{T}^n} D_{\boldsymbol{u}}^\beta f_{j_0}D_{\boldsymbol{x}}^{\alpha_0}u_i D_{\boldsymbol{x}}^{\alpha_1}u_{i_1}\cdots D_{\boldsymbol{x}}^{\alpha_{|\beta|}}u_{i_{|\beta|}}\,d\boldsymbol{x}\,,$$

where $|\alpha_0|+|\alpha_1|+\cdots+|\alpha_{|\beta|}| = k$ and the indexes i_s (where $s=1,\ldots,|\beta|$) vary between 1 and m. The modulus of this integral is not bigger than

$$\mathfrak{A} = \big|D_{\boldsymbol{u}}^\beta f_{j_0}\big|_{L_\infty(B(|\boldsymbol{u}|))}\big|D_{\boldsymbol{x}}^{\alpha_0}u_i\big|_{L_{\frac{2k}{|\alpha_0|}}}\big|D_{\boldsymbol{x}}^{\alpha_1}u_{i_1}\big|_{L_{\frac{2k}{|\alpha_1|}}}\cdots\big|D_{\boldsymbol{x}}^{\alpha_{|\beta|}}u_{i_{|\beta|}}\big|_{L_{\frac{2k}{|\alpha_{|\beta|}|}}}\,.$$

Here $B(r)$ denotes the ball in $\mathbb{R}^m$ of radius r in the Euclidean norm, centered at the origin. Using the Gagliardo-Nirenberg inequality (see [9], pp. 106-107)

$$|D_{\boldsymbol{x}}^{\alpha_s}u_{i_s}|_{L_{\frac{2k}{|\alpha_s|}}} \leqslant 4^{|\alpha_s|(k-|\alpha_s|)}|u_{i_s}|_{L_\infty}^{1-\frac{|\alpha_s|}{k}}\,\|u_{i_s}\|_k^{\frac{|\alpha_s|}{k}}\,,$$

and the inequality $\sum_{s=0}^{|\beta|}(|\alpha_s|k-|\alpha_s|^2)\leqslant\sum_{s=0}^{|\beta|}(|\alpha_s|k-|\alpha_s|)=k^2-k$ we obtain

$$\mathfrak{A}\leqslant 4^{k^2-k}|D_{\boldsymbol{u}}^{\beta}f_j|_{L_\infty(B(|\boldsymbol{u}|))}|\boldsymbol{u}|_{L_\infty}^{|\beta|}\left\|\boldsymbol{u}\right\|_k .$$

Now using the fact that left hand side of (2.9) $\leqslant(m+1)\cdots(m+k-1)\max\{\mathfrak{A}\}$, we arrive at (2.9) with $C'_{k,m,n}=4^{k^2-k}(m+1)(m+2)\cdots(m+k-1)$. □

Corollary 1. For $k\geqslant 1$ we have:

$$b(\boldsymbol{f}(\boldsymbol{u}),\boldsymbol{u},(-\Delta)^k\boldsymbol{u})\leqslant B_k(t)\left\|\boldsymbol{u}\right\|_k\left\|\boldsymbol{u}\right\|_{k+1} ,$$

where $B_k(t)=C_{k,m,n}\max_{s=0\dots k-1}\{[\boldsymbol{f}]_{C^s(|\boldsymbol{u}_0|+H(t))}(|\boldsymbol{u}_0|+H(t))^s\}$.

Integrating Υ_2^k by parts, we obtain $\Upsilon_2^k\leqslant\left\|\boldsymbol{h}\right\|_{k-1}\left\|\boldsymbol{u}\right\|_{k+1}$, So we have

$$\tfrac{1}{2}\tfrac{d}{dt}\left\|\boldsymbol{u}\right\|_k^2\leqslant\left\|\boldsymbol{u}\right\|_{k+1}(-\nu\left\|\boldsymbol{u}\right\|_{k+1}+B_k(t)\left\|\boldsymbol{u}\right\|_k+\left\|\boldsymbol{h}\right\|_{k-1}) .$$

Now using the interpolation inequality in the form $\left\|\boldsymbol{u}\right\|_{k+1}\geqslant\left\|\boldsymbol{u}\right\|_k\left(\frac{\|\boldsymbol{u}\|_k}{\|\boldsymbol{u}\|_0}\right)^{1/k}$, we have

$$\tfrac{1}{2}\tfrac{d}{dt}\left\|\boldsymbol{u}\right\|_k^2\leqslant\left\|\boldsymbol{u}\right\|_{k+1}\left(\left\|\boldsymbol{u}\right\|_k\left(-\nu\big(\tfrac{\|\boldsymbol{u}\|_k}{\|\boldsymbol{u}\|_0}\big)^{1/k}+B_k(t)\right)+\left\|\boldsymbol{h}\right\|_{k-1}\right) .$$

It follows from this relation that

$$\text{if } \left\|\boldsymbol{u}\right\|_k>\frac{(B_k(t)+1)^k\left\|\boldsymbol{u}\right\|_0}{\nu^k} \text{ and } \left\|\boldsymbol{u}\right\|_k>\left\|\boldsymbol{h}\right\|_{k-1}, \text{ then } \left\|\boldsymbol{u}\right\|_k \text{ is decreasing.} \tag{2.10}$$

We denote the right hand side of (1.6) by $F_k(t)$. It is clear from the definition of the function F_k that

$$\left\|\boldsymbol{u}(0,\cdot)\right\|_k\leqslant F_k(0).$$

Using (2.3) we see that if $\left\|\boldsymbol{u}\right\|_k>F_k(t)$ then $\left\|\boldsymbol{u}\right\|_k$ is decreasing by argument (2.10). Since $F_k(t)$ is a non-decreasing function we obtain that $\left\|\boldsymbol{u}\right\|_k$ never can be greater than $F_k(t)$. We arrive at (1.6). Theorem 1 is proven.

3 Lower estimates

In this section we prove theorem 2. Throughout this section we use standard facts from linear algebra about linear transformations. For the convenience of the reader we very briefly outline the proofs. See reference [7], for an elegant, coordinate free presentation. We start from brief discussing of the notion of the degeneracy of a vector field.

3.1 Degeneracy condition

Using the fact that an $n\times n$ matrix A is nilpotent iff $A^n=0$, we can give a definition of degeneracy that is equivalent to the previous one, but more robust.

Definition 2. *The vector field* $\boldsymbol{u}_0$ *is degenerate iff* $\left(\frac{\partial\boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{x}))}{\partial\boldsymbol{x}}\right)^n\equiv 0.$

Let $\tilde{\boldsymbol{f}}(\boldsymbol{x})=\boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{x}))$. Consider the characteristic polynomial of the matrix $\frac{\partial\tilde{\boldsymbol{f}}}{\partial\boldsymbol{x}}$:

$$\chi_{\boldsymbol{x}}(\lambda)=\det\big(\tfrac{\partial\tilde{\boldsymbol{f}}}{\partial\boldsymbol{x}}-\lambda\mathbf{1}\big)=(-\lambda)^n+(-\lambda)^{n-1}I_1(\boldsymbol{x})+\cdots+I_n(\boldsymbol{x}) .$$

Expanding the determinant, we obtain

$$I_k(\boldsymbol{x})=\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant n}\det\begin{pmatrix}\frac{\partial\tilde{f}_{i_1}}{\partial x_{i_1}} & \cdots & \frac{\partial\tilde{f}_{i_1}}{\partial x_{i_k}}\\ \vdots & \ddots & \vdots\\ \frac{\partial\tilde{f}_{i_k}}{\partial x_{i_1}} & \cdots & \frac{\partial\tilde{f}_{i_k}}{\partial x_{i_k}}\end{pmatrix} . \tag{3.1}$$

Using the Jordan form of the matrix we see that if $\frac{\partial \tilde{f}_i}{\partial x_j}(\boldsymbol{x})$ is nilpotent then all $I_j(\boldsymbol{x})$ are zero numbers. From the Hamilton-Cayley identity (any matrix is a root of its characteristic polynomial) we get the converse. Thus we have that the matrix $\frac{\partial \tilde{f}_i}{\partial x_j}(\boldsymbol{x})$ is nilpotent iff $I_1(\boldsymbol{x}) = \cdots = I_n(\boldsymbol{x}) = 0$. We got another equivalent definition of degeneracy which we will use subsequently:

Definition 3. *The vector field $\boldsymbol{u}_0$ is degenerate iff $I_k(\boldsymbol{x}) \equiv 0$ for all $k \in \{1, \ldots, n\}$.*

We also note that if $m < n$, then the matrix $\frac{\partial \tilde{f}_i}{\partial x_j}$ has rank $\leqslant m$, so for $k \in [m+1, n]$ we have $I_k(\boldsymbol{x}) \equiv 0$.

Lemma 3. *For each $k = 1, \ldots, n$ we have $\int_{\mathbb{T}^n} I_k(\boldsymbol{x}) d\boldsymbol{x} = 0$.*

Proof. We need to show that

$$\int_{\mathbb{T}^n} \det\left(\frac{\partial \tilde{f}_i}{\partial x_j}(\boldsymbol{x}) + \lambda \mathbf{1}\right) d\boldsymbol{x} \equiv (\ell\lambda)^n. \tag{3.2}$$

Since both the left hand side and the right hand side are polynomials in λ, it is sufficient to prove this equality for all integer λ. We write $\frac{\partial \tilde{f}_i}{\partial x_j}(\boldsymbol{x}) + \lambda \mathbf{1} = \frac{\partial \boldsymbol{\Psi}}{\partial \boldsymbol{x}}$, where the vector valued function $\boldsymbol{\Psi}$ is defined by the formula $\boldsymbol{\Psi}(\boldsymbol{x}) = \tilde{\boldsymbol{f}}(\boldsymbol{x}) + \lambda \boldsymbol{x}$. Since λ is an integer, then this function defines a map from $\mathbb{T}^n$ to $\mathbb{T}^n$. Since $\boldsymbol{\Psi}$ is homotopic to the map $\boldsymbol{x} \mapsto \lambda \boldsymbol{x}$ on torus $\mathbb{T}^n$ (a homotopy is given by $\boldsymbol{\Psi}_t(\boldsymbol{x}) = t\tilde{\boldsymbol{f}}(\boldsymbol{x}) + \lambda \boldsymbol{x}$), we have $\deg \boldsymbol{\Psi} = \deg\{\boldsymbol{x} \mapsto \lambda \boldsymbol{x}\}$ and hence $\deg \boldsymbol{\Psi} = \lambda^n$. Using the formula

$$\int_{\mathbb{T}^n} \det \frac{\partial \boldsymbol{\Psi}}{\partial \boldsymbol{x}} d\boldsymbol{x} = \deg \boldsymbol{\Psi} \int_{\mathbb{T}^n} d\boldsymbol{x}$$

(see [3], II, chpt. 3) we arrive at (3.2) (since $\int_{\mathbb{T}^n} d\boldsymbol{x} = \ell^n$). ☐

It follows that any potential degenerate initial state is constant. Indeed, if $\boldsymbol{f}(\boldsymbol{u}_0) = \nabla U$ then the function $U : \mathbb{R}^n \to \mathbb{R}$ necessarily has no more than linear growth (because $\boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{x}))$ is periodic) and is also a harmonic function (because $\Delta U(\boldsymbol{x}) = \operatorname{div} \boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{x})) = I_1(\boldsymbol{x}) \equiv 0$); so $U(\boldsymbol{x}) = B\boldsymbol{x} + \boldsymbol{c}$ where B and $\boldsymbol{c}$ are a constant matrix and a constant vector, respectively.

Consider the case $n = 2$, i.e., $\dim \boldsymbol{x} = 2$.

Theorem 3. *Let n=2. Then the vector field $\boldsymbol{u}_0$ is degenerate iff there exist a function $\varphi_0 : \mathbb{R} \to \mathbb{R}$ and real numbers b_1, b_2, c_1, and c_2 such that*

$$\begin{aligned} \{\boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{x}))\}_1 &= b_2 \varphi_0(b_1 x_1 + b_2 x_2) + c_1, \\ \{\boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{x}))\}_2 &= -b_1 \varphi_0(b_1 x_1 + b_2 x_2) + c_2. \end{aligned} \tag{3.3}$$

Proof. The sufficiency is trivial. Indeed, if the vector field $\boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{x}))$ has the form (3.3), then the Jacobi matrix

$$\begin{pmatrix} \partial f_1(\boldsymbol{u}_0)/\partial x_1 & \partial f_1(\boldsymbol{u}_0)/\partial x_2 \\ \partial f_2(\boldsymbol{u}_0)/\partial x_1 & \partial f_2(\boldsymbol{u}_0)/\partial x_2 \end{pmatrix} = \begin{pmatrix} b_1 b_2 \varphi_0' & b_2^2 \varphi_0' \\ -b_1^2 \varphi_0' & -b_2 b_1 \varphi_0' \end{pmatrix}$$

is nilpotent.

Necessity. Let c_1 and c_2 be the mean values of $f_1(\boldsymbol{u}_0(\boldsymbol{x}))$ and $f_2(\boldsymbol{u}_0(\boldsymbol{x}))$, respectively. Since $\operatorname{div} \boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{x})) = 0$, there exists a function $\psi : \mathbb{R}^2 \to \mathbb{R}$ such that $\operatorname{rot} \psi = \boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{x})) - \binom{c_1}{c_2}$, where $\operatorname{rot} \psi = \binom{\partial\psi/\partial x_2}{-\partial\psi/\partial x_1}$. We note that the function $\psi(x_1, x_2)$ is $\mathbb{T}^2$-periodic, and hence bounded. Since the determinant of the Jacobi matrix of $\boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{x}))$ is zero, we have that determinant of the Hessian of ψ is zero. Consider the graph of the function ψ in $\mathbb{R}^3$. The Gaussian curvature of this surface is given by the formula (see [3], I, chpt. 2)

$$K = \frac{\psi_{xx}\psi_{yy} - \psi_{xy}^2}{(1 + \psi_x^2 + \psi_y^2)^2}.$$

Therefore, $K = 0$. Now we use the fact that any complete surface of constant zero Gaussian curvature is a cylinder over a flat curve (see [13]; [14], chpt. 5). Since the function ψ is bounded, every generator of this cylinder is a horizontal line, hence it's equation can be written in the form

$$\begin{cases} b_1x_1 + b_2x_2 = const, \\ z = \tilde{\psi}(const). \end{cases}$$

We conclude that

$$\psi(x_1, x_2) = \tilde{\psi}(b_1x_1 + b_2x_2)$$

and (3.3) follows with $\varphi_0 = \tilde{\psi}'$. □

Corollary 2. *Suppose, $m = n = 2$, $\boldsymbol{f}(\boldsymbol{u}) \equiv \boldsymbol{u}$, and $\boldsymbol{h} \equiv 0$; then the solution of the Cauchy problem (1.1), (3.3) remains of the form (3.3):*

$$\boldsymbol{u}(t,\boldsymbol{x}) = \begin{pmatrix} b_2 \\ -b_1 \end{pmatrix} \varphi(t, b_1x_1 + b_2x_2) + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

where the function φ satisfies the following equation

$$\varphi_t + (b_1c_1 - b_2c_2)\varphi' = (b_1^2 + b_2^2)\nu\varphi''.$$

In this case we have ν-independent upper bounds for derivatives of the solution.

Further on we shall use the polynomial

$$P_{\boldsymbol{x}}(t) = t^n \chi_{\boldsymbol{x}}(\tfrac{-1}{t}) = \det\left(\delta_{ij} + \tfrac{\partial \tilde{f}_i}{\partial x_j} t\right) = 1 + I_1(\boldsymbol{x})t + I_2(\boldsymbol{x})t^2 + \cdots + I_n(\boldsymbol{x})t^n, \tag{3.4}$$

rather than a characteristic polynomial.

3.2 General idea

In this subsection we present an auxiliary theorem from which we then derive theorem 2. This auxiliary theorem is technically complicated. Here we deal mainly with general ideas, and postpone the technicalities to the next subsection.

We denote the right hand side of (2.1) by g_i:

$$\tfrac{\partial}{\partial t} u_i + \sum_{j=1}^{n} f_j(u_1, \ldots, u_m) \tfrac{\partial u_i}{\partial x_j} = g_i(t, x_1, \ldots, x_n)\,. \tag{3.5}$$

Theorem 4. *Let $\boldsymbol{f} \colon \mathbb{R}^m \to \mathbb{R}^n$ be a C^1-smooth map and let $\boldsymbol{u}_0 \colon \mathbb{T}^n \to \mathbb{R}^m$ be a C^1-smooth vector field.*

1) *If $\boldsymbol{u}_0$ is non-degenerate, then there exist $T = T(\boldsymbol{f}, \boldsymbol{u}_0) < \infty$ and $c = c(\boldsymbol{f}, \boldsymbol{u}_0) > 0$ such that for any C^1-smooth vector field $\boldsymbol{u} \colon [0,T] \times \mathbb{T}^n \to \mathbb{R}^m$ such that $\boldsymbol{u}(0,\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x})$ we have*

$$\int_0^T \sup_{\boldsymbol{x} \in \mathbb{T}^n} |\boldsymbol{g}(\tau, \boldsymbol{x})| d\tau \geqslant c\,, \tag{3.6}$$

where $\boldsymbol{g}$ is given by (3.5).

2) *If $\boldsymbol{u}_0$ is degenerate, then there is a C^1-smooth vector field $\boldsymbol{u} \colon [0, +\infty) \times \mathbb{T}^n \to \mathbb{R}^m$ such that $\boldsymbol{u}(0,\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x})$ and $g_i(t,\boldsymbol{x}) \equiv 0$.*

Proof. 1) Without loss of generality it can be assumed that $\boldsymbol{u}(t,\boldsymbol{x})$ is defined for all $t \geqslant 0$. Consider the flow on the cylinder $\mathbb{T}^n \times [0,\infty)$ generated by the vector field $\boldsymbol{f}(\boldsymbol{u})$. In other words we consider the Cauchy problem

$$\tfrac{\partial}{\partial t} \boldsymbol{\gamma}(t, \boldsymbol{\xi}) = \boldsymbol{f}(\boldsymbol{u}(t, \boldsymbol{\gamma}(t, \boldsymbol{\xi})))$$

with the initial state $\boldsymbol{\gamma}(0, \boldsymbol{\xi}) = \boldsymbol{\xi}$. Here $\boldsymbol{\xi}$ is the Lagrange coordinate of the flow $\boldsymbol{\gamma}$.

For any fixed time t we have a map $\boldsymbol{\gamma}(t,\cdot) : \mathbb{T}^n \to \mathbb{T}^n$. Since $\boldsymbol{\gamma}(t,\cdot)$ is a continuous family of diffeomorphisms, equal identity for $t = 0$, then its Jacobian is everywhere positive.

Combining the chain rule and (3.5), we obtain

$$\tfrac{d}{dt}\boldsymbol{u}(t,\boldsymbol{\gamma}(t,\boldsymbol{\xi})) = \boldsymbol{g}(t,\boldsymbol{\gamma}(t,\boldsymbol{\xi}))\,.$$

Suppose $\boldsymbol{g}(\cdot) \equiv 0$; then $\boldsymbol{f}\big(\boldsymbol{u}(t,\boldsymbol{\gamma}(t,\boldsymbol{\xi}))\big) \equiv \boldsymbol{f}\big(\boldsymbol{u}(0,\boldsymbol{\gamma}(0,\boldsymbol{\xi}))\big)$ and $\boldsymbol{\gamma}(t,\boldsymbol{\xi}) = \boldsymbol{\gamma^0}(t,\boldsymbol{\xi})$, where

$$\boldsymbol{\gamma^0}(t,\boldsymbol{\xi}) = \boldsymbol{\xi} + t\boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{\xi}))\,. \tag{3.7}$$

It follows that if the function $\boldsymbol{g}$ is small, then the flow $\boldsymbol{\gamma}(t,\boldsymbol{\xi})$ is close (in the C^0-norm) to the map (3.7). For a detailed proof of this fact, we refer to the next subsection. For the time being we simply note that this is a consequence of the following inequality:

$$\big|\boldsymbol{u}(t,\boldsymbol{\gamma}(t,\boldsymbol{\xi})) - \boldsymbol{u}(0,\boldsymbol{\gamma}(0,\boldsymbol{\xi}))\big| \leqslant \int_0^t \sup_{\boldsymbol{x}\in\mathbb{T}^n} |\boldsymbol{g}(\tau,\boldsymbol{x})|d\tau\,.$$

Since for each $k = 1,\dots,n$, we have $\int_{\mathbb{T}^n} I_k(\boldsymbol{x})d\boldsymbol{x} = 0$ (see lemma 3) and since some of the I_k are not identically zero (due to the non-degeneracy of $\boldsymbol{u}_0$), we obtain that there exists a point $\boldsymbol{x}^\star \in \mathbb{T}^n$ and a number $l \in [1,\dots,n]$ such that $I_l(\boldsymbol{x}^\star) < 0$ and $I_k(\boldsymbol{x}^\star) = 0$ for $k > l$.

The Jacobian of (3.7) is expressed by polynomial (3.4). For the time $t = T$ at the point $\boldsymbol{\xi} = \boldsymbol{x}^\star$, we have

$$\det\big(\tfrac{\partial\boldsymbol{\gamma}^0}{\partial\boldsymbol{\xi}}\big|_{\substack{t=T\\ \boldsymbol{\xi}=\boldsymbol{x}^\star}}\big) = P_{\boldsymbol{x}^\star}(T) = 1 + I_1(\boldsymbol{x}^\star)T + I_2(\boldsymbol{x}^\star)T^2 + \dots + I_l(\boldsymbol{x}^\star)T^l\,.$$

We take large enough time T such that this Jacobian is negative. Suppose that

$$\int_0^T \sup_{\boldsymbol{x}\in\mathbb{T}^n} |\boldsymbol{g}(\tau,\boldsymbol{x})|d\tau < c,$$

where c is a small enough number; then the map $\boldsymbol{\gamma}(T,\cdot)$ is close to the map $\boldsymbol{\gamma}^0(T,\cdot)$ with a negative Jacobian at the point $\boldsymbol{x}^\star$. Taking c small enough we have a contradiction with the positivity of the Jacobian of the map $\boldsymbol{\gamma}(T,\cdot)$. (See the next part of this section for more details).

2) Consider the map (3.7). For any fixed t we have a (C^1-smooth) map $\boldsymbol{\gamma^0}(t,\cdot)\colon \mathbb{T}^n \to \mathbb{T}^n$. We note that this map is a (C^1-smooth) diffeomorphism of the torus $\mathbb{T}^n$ iff the Jacobian of $\boldsymbol{\gamma^0}(t,\cdot)$ is everywhere positive. Indeed, if this map is a diffeomorphism, then the Jacobian is not vanishing, hence it has the same sign for all points, and this sign is positive since the map is homotopic to the identity map. If the Jacobian is everywhere positive then by the inverse function theorem we have that $\boldsymbol{\gamma^0}(t,\cdot)$ is a (local) diffeomorphism in a neighbourhood of any point. Since the Jacobian is everywhere positive, the number of preimages of any point $\boldsymbol{z}$ is finite and equals the degree of the map. (see [3], II, chpt. 3). On the other hand, the degree of the map $\boldsymbol{\gamma^0}(t,\cdot)$ is equal to 1, since this map is homotopic to the identical map $\boldsymbol{\gamma^0}(0,\cdot)$. (see [3], II, chpt. 3). Hence each point $\boldsymbol{z}$ has a unique pre-image $(\boldsymbol{\gamma^0})^{-1}(t,\boldsymbol{z})$.

The Jacobian of the map $\boldsymbol{\gamma^0}(t,\cdot)$ at a point $\boldsymbol{x}$ is expressed by the polynomial (3.4):

$$\big(\tfrac{\partial\boldsymbol{\gamma}^0}{\partial\boldsymbol{\xi}}\big) = P_{\boldsymbol{x}}(t).$$

If $\boldsymbol{u}_0$ is degenerate then $P_{\boldsymbol{x}}(t) \equiv 1$. Hence the vector field

$$\boldsymbol{u}(t,\boldsymbol{x}) = \boldsymbol{u}_0\big((\boldsymbol{\gamma^0})^{-1}(t,\boldsymbol{x})\big)$$

is well defined and satisfies $\partial_t\boldsymbol{u} + \nabla_{\boldsymbol{f}(\boldsymbol{u})}\boldsymbol{u} = 0$ as the second part of theorem 4 states. □

Let us turn to the proof of the theorem 2. Let $\boldsymbol{u}$ satisfies equation (1.1). Suppose that $H(t) < \frac{c}{2}$, where $H(t)$ is defined by (1.3); then we obtain

$$\int_0^T \sup_{\boldsymbol{x}\in\mathbb{T}^n} |\nu\Delta\boldsymbol{u}(\tau,\boldsymbol{x})|\,d\tau > \frac{c}{2}\,.$$

Hence we have inequality (1.7) for $k = 2$ with the constant $r_2 = \frac{c}{2nT}$. We now fix the index j to the value for which the maximum in (1.7) for $k = 2$ is achieved. For this index we have

$$\frac{1}{T}\int_0^T \sup_{\boldsymbol{x}\in\mathbb{T}^n} \left|\tfrac{\partial^2 \boldsymbol{u}}{\partial x_j^2}(\tau,\boldsymbol{x})\right| d\tau \geqslant \tfrac{c}{2nT}\,\tfrac{1}{\nu}\,. \tag{3.8}$$

To complete the proof we need the interpolation inequality

$$\left|\tfrac{\partial^2 \boldsymbol{u}}{\partial x_j^2}(\tau,\cdot)\right|_{L_\infty} \leqslant C_{k,2}\,|\boldsymbol{u}(\tau,\cdot)|_{L_\infty}^{\frac{k-2}{k}}\,\left|\tfrac{\partial^k \boldsymbol{u}}{\partial x_j^k}(\tau,\cdot)\right|_{L_\infty}^{\frac{2}{k}}. \tag{3.9}$$

For its proof with the best possible constants for the $1D$ case ($m = n = 1$) see [10]. The case of arbitrary dimensions can be reduced to the $1D$ case by considering the function $v(x_j) = \sum_{i=1}^m a_i u_i(x_1^0,\dots,x_{j-1}^0,x_j,x_{j+1}^0,\dots,x_n^0)$, where $\boldsymbol{x^0} = \boldsymbol{x^0}(\tau) \in \mathbb{T}^n$ is the maximum point of the $\left|\tfrac{\partial^2 \boldsymbol{u}}{\partial x_j^2}(\tau,\boldsymbol{x})\right|$ and $\boldsymbol{a}$ is a constant unit vector in $\mathbb{R}^m$, proportional to $\tfrac{\partial^2 \boldsymbol{u}}{\partial x_j^2}(\tau,\boldsymbol{x^0})$. We note that this reduction preserves the Kolmogorov's constants.

Using (3.9) and inequality (2.2), we obtain

$$\left|\tfrac{\partial^k \boldsymbol{u}}{\partial x_j^k}(\tau,\cdot)\right|_{L_\infty} \geqslant \frac{\left|\tfrac{\partial^2 \boldsymbol{u}}{\partial x_j^2}(\tau,\cdot)\right|_{L_\infty}^{k/2}}{\sqrt{3}\big(|\boldsymbol{u}_0| + H(\tau)\big)^{\frac{k-2}{2}}}\,. \tag{3.10}$$

Here we have used the inequality $(C_{k,2})^{k/2} \leqslant \sqrt{3}$, which can be easily proved using Kolmogorov's explicit representation (see [10]) via the Hölder inequality. Integrating (3.10) we obtain

$$\frac{1}{T}\int_0^T \left|\tfrac{\partial^k \boldsymbol{u}}{\partial x_j^k}(\tau,\cdot)\right|_{L_\infty} d\tau \geqslant \frac{\frac{1}{T}\int_0^T \left|\tfrac{\partial^2 \boldsymbol{u}}{\partial x_j^2}(\tau,\cdot)\right|_{L_\infty}^{k/2} d\tau}{\sqrt{3}\big(|\boldsymbol{u}_0| + H(T)\big)^{\frac{k-2}{2}}} \geqslant$$
$$\frac{\left(\frac{1}{T}\int_0^T \left|\tfrac{\partial^2 \boldsymbol{u}}{\partial x_j^2}(\tau,\cdot)\right|_{L_\infty} d\tau\right)^{k/2}}{\sqrt{3}\big(|\boldsymbol{u}_0| + H(T)\big)^{\frac{k-2}{2}}} \overset{(3.8)}{\geqslant} \frac{\left(\frac{c}{2nT\nu}\right)^{k/2}}{\sqrt{3}\big(|\boldsymbol{u}_0| + H(T)\big)^{\frac{k-2}{2}}}\,.$$

This concludes the proof of (1.7) for $k > 2$ with the constants

$$r_k = \frac{\left(\frac{c}{2nT}\right)^{\frac{k}{2}}}{\sqrt{3}\big(|\boldsymbol{u}_0| + H(T)\big)^{\frac{k-2}{2}}}\,. \tag{3.11}$$

In the next subsection we will specify the values of T and c (see (3.16) and (3.33) respectively).

3.3 Technicalities

In this subsection we introduce a more general approach to the estimates of theorem 2 which applies to the non-periodic case. From the previous subsection we already know that the crucial condition for theorem 2 is the "negativity" rather than the non-degeneracy of the matrix $\frac{\partial \boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{x}))}{\partial \boldsymbol{x}}$. Let $\boldsymbol{u} : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^m$ be a C^1-smooth vector-valued function and $\boldsymbol{f} : \mathbb{R}^m \to \mathbb{R}^n$ be a C^1-smooth map. In this subsection $\boldsymbol{x} = (x_1,\dots,x_n)$ are coordinates in $\mathbb{R}^n$.

We define $g_i : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$, $(i = 1,\dots,m)$ to satisfy:

$$\partial_t u_i + \sum_{j=1}^n f_j(u_1,\dots,u_m)\frac{\partial u_i}{\partial x_j} = g_i(t,x_1,\dots,x_n)\,. \tag{3.12}$$

Let $\tilde{\boldsymbol{f}}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{u}(0,\boldsymbol{x}))$. If this function is C^2-smooth then we consider the norm

$$|\tilde{\boldsymbol{f}}|_2^2 = \sup_{\boldsymbol{x}} \sum_{i,j,k=1}^n \left|\frac{\partial^2 \tilde{f}_i}{\partial x_j \partial x_k}\right|^2. \tag{3.13}$$

For any $\boldsymbol{u} \in \mathbb{R}^m$ the derivative $\boldsymbol{\nabla_u f} : \mathbb{R}^m \to \mathbb{R}^n$ is a linear map. For any domain $E \subset \mathbb{R}^m$ we denote

$$\|\boldsymbol{\nabla_u f}\|_E = \sup_{\boldsymbol{u}\in E} \|\boldsymbol{\nabla_u f}\| = \sup_{\boldsymbol{u}\in E} \max_{|\boldsymbol{v}|=1} \big|(\boldsymbol{\nabla_u f})\boldsymbol{v}\big|, \tag{3.14}$$

where $\{(\boldsymbol{\nabla_u f})\boldsymbol{v}\}_j = \sum\limits_{i=1\ldots m} \frac{\partial f_j}{\partial u_i} v_i$. We note that for the Burgers' (=NS') nonlinearity (i.e., $m = n$ and $\boldsymbol{f} \equiv \boldsymbol{u}$) we have $\|\boldsymbol{\nabla_u f}\| \equiv 1$.

Theorem 5. *Suppose that $\boldsymbol{u} : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^m$ and $\boldsymbol{f} : \mathbb{R}^m \to \mathbb{R}^n$ are C^1-smooth. Let ε be a positive real number and $l \in \{1, 2, \ldots, n\}$. Suppose that there exists $\boldsymbol{x}^\star \in \mathbb{R}^n$ such that $I_l(\boldsymbol{x}^\star) = -\varepsilon < 0$ and that for $k = l+1, \ldots, n$ we have $I_k(\boldsymbol{x}^\star) = 0$, where I_i are defined in (3.1). Let*

$$|\tilde{\boldsymbol{f}}|_1 = \max_{i,j} \left|\frac{\partial \tilde{f}_i}{\partial x_j}(\boldsymbol{x}^\star)\right|, \tag{3.15}$$

where $\tilde{\boldsymbol{f}}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{u}(0, \boldsymbol{x}))$ and let

$$T = 2^n l^{l/2} \frac{|\tilde{\boldsymbol{f}}|_1^{l-1}}{\varepsilon}\,. \tag{3.16}$$

Then there exists a positive function $c_2(c_1)$ such that if

$$\int_0^T \sup_{\boldsymbol{x}\in\mathbb{R}^n} |\boldsymbol{g}(\tau, \boldsymbol{x})|\, d\tau < c_1\,, \tag{3.17}$$

then

$$\int_0^T (T-\tau) \sup_{\boldsymbol{x}\in\mathbb{R}^n} |\boldsymbol{g}(\tau, \boldsymbol{x})|\, d\tau \geqslant c_2(c_1)\,. \tag{3.18}$$

If the function $\tilde{\boldsymbol{f}}$ is C^2-smooth and the norm (3.13) is finite, then one can take

$$c_2(c_1) = \frac{\frac{1}{4}\, l^l}{n^{2n-2}(|\tilde{\boldsymbol{f}}|_1 T)^{2n-2l}\, \|\boldsymbol{\nabla_u f}\|_{B(|\boldsymbol{u}_0|+c_1)} |\tilde{\boldsymbol{f}}|_2 T}\,. \tag{3.19}$$

Here $B(r)$ denotes the ball in $\mathbb{R}^m$ of radius r in the Euclidean norm, centered at the origin.

Proof. First of all we note that $\boldsymbol{f}(\boldsymbol{u}(t, \boldsymbol{x}))$ and $\frac{\partial}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{u}(t, \boldsymbol{x}))$ are a continuous vector function and a continuous matrix function, respectively. It follows from this that

1° ∃! solution of the Cauchy problem for the following ODE in R^n:

$$\tfrac{d}{dt}\boldsymbol{\gamma}(t, \boldsymbol{\xi}) = \boldsymbol{f}(\boldsymbol{u}(t, \boldsymbol{\gamma}(t, \boldsymbol{\xi}))) \tag{3.20}$$

with the initial state $\boldsymbol{\gamma}(0, \boldsymbol{\xi}) = \boldsymbol{\xi} \in \mathbb{R}^n$.

2° This solution $\boldsymbol{\gamma} \in C^1\big([0, T] \times \mathbb{R}^n; \mathbb{R}^n\big)$.

3° This solution satisfies $\quad det\left(\frac{\partial \boldsymbol{\gamma}(t,\boldsymbol{\xi})}{\partial \boldsymbol{\xi}}\right) = \exp \int_0^t \sum_{i=0}^n \frac{\partial}{\partial \gamma_i} f_i\big(\boldsymbol{u}(t, \boldsymbol{\gamma}(t, \boldsymbol{\xi}))\big)$, see [8].

The positiveness of the Jacobian in 3° implies inequality (3.18). The rest of this subsection is devoted to proving this fact.

In the proof of proposition 1 bellow we will use the fact that there exists a continuous second derivative $\frac{d^2}{dt^2}\boldsymbol{\gamma}(t, \boldsymbol{\xi})$. This follows from the existence and continuity of the first partial derivative $\frac{\partial}{\partial t}\boldsymbol{f}(\boldsymbol{u}(t, \boldsymbol{x}))$.

For the quantities (3.1) we have:

$$|I_k| \leqslant \tbinom{n}{k} k^{k/2} |\tilde{\boldsymbol{f}}|_1^k\,. \tag{3.21}$$

Indeed, the right hand side of (3.1) contains $\binom{n}{k}$ terms and each of them is no greater than $k^{k/2}|\tilde{\boldsymbol{f}}|_1^k$. Here we have used the fact that the volume of a k-dimensional parallelepiped with sides of length less than or equal to $\sqrt{k}|\tilde{\boldsymbol{f}}|_1$ is no greater than $(\sqrt{k}|\tilde{\boldsymbol{f}}|_1)^k$.

In the proof of propositions 2 and 3 below we will use the inequality

$$T|\tilde{\boldsymbol{f}}|_1 > 1\,. \tag{3.22}$$

It follows from (3.16) and (3.21) with $k = l$ since $\varepsilon = |I_l(\boldsymbol{x}^\star)|$.

We fix T at the value given in (3.16). For any t, $\boldsymbol{\gamma}(t,\boldsymbol{x})$ defines a mapping from $\mathbb{R}^n$ into itself. We take $t = T$ and decompose this mapping as follows:

$$\gamma_i(T,\boldsymbol{x}) = p_i(\boldsymbol{x}) + q_i(\boldsymbol{x})\,, \qquad i = 1,\dots,n \tag{3.23}$$

Here $\boldsymbol{p}(\boldsymbol{x})$ comprises the zeroth and the first terms of the Tailor expansion:

$$p_i(\boldsymbol{x}) = \gamma_i(0,\boldsymbol{x}) + \frac{d}{dt}\gamma_i(0,\boldsymbol{x})T = \boldsymbol{x} + \tilde{\boldsymbol{f}}(\boldsymbol{x})T\,. \tag{3.24}$$

The remainder of the Tailor expansion can be represented as

$$q_i(\boldsymbol{x}) = \int_0^T (T-\tau)\frac{d^2}{dt^2}\gamma_i(\tau,\boldsymbol{x})\,d\tau\,. \tag{3.25}$$

Proposition 1. *For the Euclidean norm of the vector $\boldsymbol{q}$ we have:*

$$|\boldsymbol{q}| \leqslant \|\boldsymbol{\nabla_u f}\|_{B(|\boldsymbol{u}_0|+c_1)} \int_0^T (T-\tau)\sup_{\boldsymbol{x}\in\mathbb{R}^n} |\boldsymbol{g}(\tau,\boldsymbol{x})|\,d\tau\,. \tag{3.26}$$

We recall that $|\cdot|$ denotes the Euclidean norm.

Proof. Combining the chain rule with (3.20) and (3.12) we have

$$\frac{d}{dt}u_i(t,\boldsymbol{\gamma}(t,\boldsymbol{x})) = g_i(t,\boldsymbol{\gamma}(t,\boldsymbol{x}))\,.$$

Using this equality and assumption (3.17) we obtain that the function $\boldsymbol{u}$ takes values in the ball $B(|\boldsymbol{u}_0| + c_1)$ if $t \leqslant T$. We calculate the second derivative of γ_i (for $i = 1,\dots,n$):

$$\frac{d^2}{dt^2}\gamma_i(\tau,\boldsymbol{x}) = \frac{d}{dt}f_i(\boldsymbol{u}(t,\boldsymbol{\gamma}(t,\boldsymbol{x})))\Big|_{t=\tau} = \sum_{j=1}^m \frac{\partial f_i}{\partial u_j} g_j(\tau,\boldsymbol{\gamma}(\tau,\boldsymbol{x})) = \{(\boldsymbol{\nabla_u f})\boldsymbol{g}\}_i\,.$$

From this formula we obtain

$$\left|\tfrac{d^2}{dt^2}\boldsymbol{\gamma}(\tau,\boldsymbol{x})\right| \leqslant \|\boldsymbol{\nabla_u f}\|_{B(|\boldsymbol{u}_0|+c_1)} \sup_{\boldsymbol{x}\in\mathbb{R}^n} |\boldsymbol{g}(\tau,\boldsymbol{x})|.$$

Multiplying this inequality by $T-\tau$ and using (3.25) we arrive at (3.26). □

Consider the linearization of the map $\boldsymbol{p}$ at the point $\boldsymbol{x}^\star$ (we recall that $\boldsymbol{x}^\star$ is the point where the leading non zero I_k is negative):

$$p_i(\boldsymbol{x}) = p_i(\boldsymbol{x}^\star) + \sum_{j=1}^n \frac{\partial p_i}{\partial x_j}(\boldsymbol{x}^\star)\vartriangle x_j + \phi_i(\vartriangle\boldsymbol{x})\,, \tag{3.27}$$

where $\vartriangle\boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}^\star$ and $\boldsymbol{\phi}(\vartriangle\boldsymbol{x}) = o(\vartriangle\boldsymbol{x})$.

We need to investigate the matrix

$$A = \tfrac{\partial p_i}{\partial x_j}\Big|_{\boldsymbol{x}=\boldsymbol{x}^\star} = \delta_{ij} + T\tfrac{\partial \tilde{f}_i}{\partial x_j}\Big|_{\boldsymbol{x}=\boldsymbol{x}^\star}, \tag{3.28}$$

which is the linear part of the right hand side of (3.27).

Proposition 2. *The determinant of the matrix A is negative and bounded away from zero:*

$$det\,A \leqslant -2^{n-1}l^{l/2}(T|\tilde{\boldsymbol{f}}|_1)^{l-1}. \tag{3.29}$$

Proof. The determinant is expressed by polynomial (3.4): $det\, A = P_{\boldsymbol{x}^\star}(T)$. Using (3.21) we have

$$det\, A \leqslant 1 + \tbinom{n}{1} 1^{1/2} T|\tilde{\boldsymbol{f}}|_1 + \dots + \tbinom{n}{l-1}(l-1)^{(l-1)/2}(T|\tilde{\boldsymbol{f}}|_1)^{l-1} - \varepsilon T^l \overset{(*)}{\leqslant}$$

$$\tfrac{1}{2}2^n l^{l/2}(T|\tilde{\boldsymbol{f}}|_1)^{l-1} - \varepsilon T^l = T^{l-1}\big(\tfrac{1}{2}2^n l^{l/2}|\tilde{\boldsymbol{f}}|_1^{l-1} - \varepsilon T\big).$$

Using (3.16) we arrive at (3.29). It remains to explain inequality $(*)$. For $l = 1$ it follows from the trivial inequality $1 \leqslant 2^{n-1}$. For $l \geqslant 2$ we use the simple fact $(l-1)^{(l-1)/2} \leqslant \frac{1}{2} l^{l/2}$ and inequality (3.22) to get

$$1 + \tbinom{n}{1} 1^{1/2} T|\tilde{\boldsymbol{f}}|_1 + \dots + \tbinom{n}{l-1}(l-1)^{(l-1)/2}(T|\tilde{\boldsymbol{f}}|_1)^{l-1} \leqslant$$
$$\Big(\tbinom{n}{0} + \tbinom{n}{1} + \dots + \tbinom{n}{l-1}\Big)(l-1)^{(l-1)/2}(T|\tilde{\boldsymbol{f}}|_1)^{l-1} \leqslant 2^n \tfrac{1}{2} l^{l/2}(T|\tilde{\boldsymbol{f}}|_1)^{l-1}.$$

□

Proposition 3. *With the matrix norm $\|\cdot\|$ we have*

$$\left\|A^{-1}\right\|^{-1} \geqslant \frac{l^{l/2}}{n^{n-1}(T|\tilde{\boldsymbol{f}}|_1)^{n-l}}\,. \tag{3.30}$$

Proof. Since the numbers $\left\|A^{-1}\right\|^{-2}$ and $\|A\|^2$ are, respectively, the minimal and maximal eigenvalues of the matrix $A^t A$, we have $\left\|A^{-1}\right\|^{-1} \geqslant |\det A|\,\|A\|^{1-n}$. Using the inequalities $\|A\| \leqslant n \max_{i,j}|A_{ij}|$, $|A_{ij}| \leqslant 2T|\tilde{\boldsymbol{f}}|_1$ (the second inequality follows by (3.28) and (3.22)) and (3.29) we arrive at (3.30). □

Since $\phi_i(\vartriangle\boldsymbol{x}) = o(\vartriangle\boldsymbol{x})$, where $\phi_i(\vartriangle\boldsymbol{x})$ is the remainder term in (3.27), there exists $r > 0$ such that

$$|\phi_i(\vartriangle\boldsymbol{x})| \leqslant \tfrac{1}{2}\left\|A^{-1}\right\|^{-1} r \qquad \text{for } |\vartriangle\boldsymbol{x}| \leqslant r. \tag{3.31}$$

Consider the sphere $S_r(\boldsymbol{x}^\star)$ with the centre at the point $\boldsymbol{x}^\star$ and with the radius r.

Proposition 4. *There exists $\boldsymbol{x_0} \in S_r(\boldsymbol{x}^\star)$ such that $|q(\boldsymbol{x_0})| \geqslant \frac{1}{2}\left\|A^{-1}\right\|^{-1} r$.*

Proof. Suppose that $|\boldsymbol{q}| < \frac{1}{2}\left\|A^{-1}\right\|^{-1} r$ on $S_r(\boldsymbol{x}^\star)$. Let $\boldsymbol{\rho}(\vartriangle\boldsymbol{x}) = \boldsymbol{\phi}(\vartriangle\boldsymbol{x}) + \boldsymbol{q}(\boldsymbol{x}^\star + \vartriangle\boldsymbol{x})$. Then due to (3.23) and (3.27) we obtain:

$$\boldsymbol{\gamma}(T, \boldsymbol{x}^\star + \vartriangle\boldsymbol{x}) = \boldsymbol{p}(\boldsymbol{x}^\star) + A\vartriangle\boldsymbol{x} + \boldsymbol{\rho}(\vartriangle\boldsymbol{x})\,.$$

We recall that $\boldsymbol{x}^\star + \vartriangle\boldsymbol{x} = \boldsymbol{x}$. Using the inequality $|\boldsymbol{\rho}(\vartriangle\boldsymbol{x})| < \left\|A^{-1}\right\|^{-1} r$ and (3.30), we have

$$|A\vartriangle\boldsymbol{x}| \geqslant \left\|A^{-1}\right\|^{-1}|\vartriangle\boldsymbol{x}| > |\boldsymbol{\rho}(\vartriangle\boldsymbol{x})| \qquad \text{for } |\vartriangle\boldsymbol{x}| = r \text{ , i.e., } \boldsymbol{x} \in S_r(\boldsymbol{x}^\star)\,.$$

From this inequality it follows that the Gauss spherical map $\Gamma : S_r(\boldsymbol{x}^\star) \to S_1(\boldsymbol{0})$

$$\vartriangle\boldsymbol{x} \mapsto \frac{A\vartriangle\boldsymbol{x} + \boldsymbol{\rho}(\vartriangle\boldsymbol{x})}{|A\vartriangle\boldsymbol{x} + \boldsymbol{\rho}(\vartriangle\boldsymbol{x})|} \tag{3.32}$$

is well defined and is homotopic to the map

$$\vartriangle\boldsymbol{x} \mapsto \frac{A\vartriangle\boldsymbol{x}}{|A\vartriangle\boldsymbol{x}|}\,.$$

Hence the degrees of these maps coincide and are equal to sign $det\, A = -1$. (see [3], II, chpt. 3). On the other hand, the map (3.32) can be written as

$$\vartriangle\boldsymbol{x} \mapsto \frac{\boldsymbol{\gamma}(\boldsymbol{x}^\star + \vartriangle\boldsymbol{x}) - \boldsymbol{p}(\boldsymbol{x}^\star)}{|\boldsymbol{\gamma}(\boldsymbol{x}^\star + \vartriangle\boldsymbol{x}) - \boldsymbol{p}(\boldsymbol{x}^\star)|}\,.$$

Since the Jacobian of $\boldsymbol{\gamma}$ does not vanish, then the degree of this map is equal to

$$\sum_{\substack{\boldsymbol{y}\in\boldsymbol{\gamma}^{-1}(\boldsymbol{p}(\boldsymbol{x}^\star))\\ \boldsymbol{y}\in B(r)+\boldsymbol{x}^\star}} \operatorname{sign} det\left(\frac{\partial\boldsymbol{\gamma}(T,\boldsymbol{\xi})}{\partial\boldsymbol{\xi}}\Big|_{\boldsymbol{\xi}=\boldsymbol{y}}\right)$$

(see [3], II, chpt. 3). This number is nonnegative, so we got a contradiction which proves proposition 4. □

Using (3.26) we arrive at (3.18) with $c_2(c_1) = \frac{\left\|A^{-1}\right\|^{-1} r}{2\,\|\nabla_{\boldsymbol{u}}\boldsymbol{f}\|_{B(|\boldsymbol{u}_0|+c_1)}}$.

Suppose that the function $\tilde{\boldsymbol{f}}$ is C^2-smooth and the norm (3.13) is finite. Then the remainder term $\phi_i(\vartriangle\boldsymbol{x})$ can be written as

$$\phi_i(\vartriangle\boldsymbol{x}) = \int_0^1 (1-\theta)\sum_{j,k=1}^{n}\frac{\partial^2 p_i}{\partial x_j\partial x_k}(\boldsymbol{x}^\star+\theta\vartriangle\boldsymbol{x})\vartriangle x_j\vartriangle x_k d\theta.$$

Hence

$$\sum_{i=1}^{n}\phi_i^2(\vartriangle\boldsymbol{x}) = \sum_{i=1}^{n}\Big(\sum_{j,k=1}^{n}\int_0^1(1-\theta)\frac{\partial^2 p_i}{\partial x_j\partial x_k}(\boldsymbol{x}^\star+\theta\vartriangle\boldsymbol{x})d\theta\vartriangle x_j\vartriangle x_k\Big)^2 \leqslant$$
$$\int_0^1(1-\theta)^2\sum_{i,j,k=1}^{n}\Big(\frac{\partial^2 p_i}{\partial x_j\partial x_k}(\boldsymbol{x}^\star+\theta\vartriangle\boldsymbol{x})\Big)^2 d\theta\sum_{j,k=1}^{n}(\vartriangle x_j\vartriangle x_k)^2 \leqslant \tfrac{1}{3}|\tilde{\boldsymbol{f}}|_2^2T^2\,|\vartriangle\boldsymbol{x}|^4.$$

Now we see that (3.31) holds with

$$r = \frac{\left\|A^{-1}\right\|^{-1}}{2|\tilde{\boldsymbol{f}}|_2T}.$$

and (3.19) follows. Theorem 5 is proved. □

Using the inequality

$$\int_0^T \sup|\boldsymbol{g}(\tau,\cdot)|\,d\tau \geqslant \frac{1}{T}\int_0^T(T-\tau)\sup|\boldsymbol{g}(\tau,\cdot)|\,d\tau$$

we obtain $\int_0^T\sup|\boldsymbol{g}(\tau,\cdot)|\,d\tau \geqslant c$ with

$$c = \sup_{c_1>0}\min\left\{c_1, \tfrac{c_2(c_1)}{T}\right\}. \tag{3.33}$$

Let $\boldsymbol{u}(0,\boldsymbol{x})$ be periodic with compact fundamental periodic domain $\mathbb{T}$. It follows that $\int_{\mathbb{T}} I_k(\boldsymbol{x})d\boldsymbol{x} = 0$ (see lemma 3); so our theorem is applicable for the periodic case iff not all $I_k(\boldsymbol{x})$ are identically zero. In this case we can put $l = \max\{k\in 1,2,\dots,n : I_k\not\equiv 0\}$ and $\varepsilon = -\min I_l(\boldsymbol{x})$.

Hence, we have proved theorem 2 with r_k as in (3.11), and c as in (3.33).

4 Fourier coefficients

In this section we present some results concerning behaviour of the Fourier coefficients of solutions for equation (1.1) which follow from what we have proved in the previous sections. These results are consistent with the so-called Kolmogorov–Obukhov (K-O) spectral asymptotics.

The K-O spectral law concerns distribution of the Fourier coefficients $\widehat{\boldsymbol{v}}_{\boldsymbol{s}}(t) = \frac{1}{\ell^{n/2}}\int_{\mathbb{T}^n}\boldsymbol{v}(t,\boldsymbol{x})e^{\frac{-i2\pi \boldsymbol{s}\boldsymbol{x}}{\ell}}d\boldsymbol{x}$ of a velocity field $\boldsymbol{v}(t,\boldsymbol{x})$ which describes turbulent motion of $3D$ fluid with small viscosity. Due to the law, there exist non-negative constants $\kappa_1<\kappa_2$ and $\varkappa$ such that for $\left(\frac{1}{\nu}\right)^{\kappa_1} < |\boldsymbol{s}| < \left(\frac{1}{\nu}\right)^{\kappa_2}$ (the inertial range) we have $\langle|\widehat{\boldsymbol{v}}_{\boldsymbol{s}}|^2\rangle \sim \left(\frac{1}{|\boldsymbol{s}|}\right)^{\varkappa+n-1}$, i.e., the energy supported by wave-numbers $\boldsymbol{v_s}$ on the sphere $\{r - Const \leqslant |\boldsymbol{s}| \leqslant r + Const\}$ behaves as $\left(\frac{1}{r}\right)^{\varkappa}$. Here $\langle\cdot\rangle$ denotes averaging over time and over a band of wave vectors. For $|\boldsymbol{s}| > \left(\frac{1}{\nu}\right)^{\kappa_2}$ (the dissipation range) the quantities $|\widehat{\boldsymbol{v}}_{\boldsymbol{s}}|$ decay faster than any power of $\frac{1}{|\boldsymbol{s}|}$. The theory does not say much about the energy range $|\boldsymbol{s}| < \left(\frac{1}{\nu}\right)^{\kappa_1}$ (see [5], Chpts 5, 6). The quantity ν^{κ_2} is called the Kolmogorov dissipation scale and $\varkappa$ is called the exponent of the K-O law.

Here and subsequently, $|\boldsymbol{s}|$ stands for the Euclidean norm of an integer vector $\boldsymbol{s} \in \mathbb{Z}^n$.

The K-O law is an *heuristic* law which applies to motion of $3D$ fluid, i.e., to solutions of the $3D$ Navier-Stokes system. Below we prove some rigorous results for solutions of the generalised Burgers equation 1.1, which is a K-O *type* spectral law. Roughly, we show that for time-averaged squared Fourier coefficients of the solutions we have $\kappa_2 \in [\frac{1}{2}, 1]$ and $\varkappa > 1$. Under the additional assumption $\kappa_1 < \frac{1}{[\frac{n}{2}]+3}$ we obtain the upper bound for the exponent of the spectral law: $\varkappa \leqslant 2[\frac{n}{2}] + 5$. Under the assumption $\kappa_1 < \frac{1}{2}$ we have $\varkappa \leqslant \varkappa(\kappa_1)$, but our estimate $\varkappa(p)$ blows up to infinity as $p \to \frac{1}{2}$.

In [1] it is shown that for the 1D case the Kolmogorov dissipation scale is equal to ν (i.e., $\kappa_2 = 1$) and the exponent of the spectral law is equal to 2.

4.1 General situation

In this subsection we prove general lemmas, which provide information on the Fourier coefficients, if we know upper and lower bounds for the Sobolev norms. First steps in this direction were made by Kuksin [12] and most of the ideas in this section actually guided by the [12]. However we are adopt here a slightly different presentation and present some alternative proofs.

Lemma 4. *Suppose that there exist real numbers $k \geqslant 0$, $p''(k)$, $c''_k > 0$ and a set $\Upsilon''_k \subset (0, 1]$ such that for any $\nu \in \Upsilon''_k$ we have*

$$\sum_{\boldsymbol{s}\in\mathbb{Z}^n} |\boldsymbol{s}|^{2k} \hat{a}^2_{\boldsymbol{s}}(\nu) \leqslant c''_k \left(\tfrac{1}{\nu}\right)^{2p''(k)}. \tag{4.1}$$

Then for any positive real number y and any $\nu \in \Upsilon''_k$ we have

$$\sum_{|\boldsymbol{s}|\geqslant y} \hat{a}^2_{\boldsymbol{s}}(\nu) \leqslant c''_k y^{-2k} \nu^{-2p''(k)}. \tag{4.2}$$

Proof. For any positive real y we have

$$\sum_{|\boldsymbol{s}|\geqslant y} \hat{a}^2_{\boldsymbol{s}} \leqslant y^{-2k} \sum_{|\boldsymbol{s}|\geqslant y} |\boldsymbol{s}|^{2k} \hat{a}^2_{\boldsymbol{s}} \leqslant y^{-2k} \sum_{\boldsymbol{s}\in\mathbb{Z}^n} |\boldsymbol{s}|^{2k} \hat{a}^2_{\boldsymbol{s}}.$$

Using (4.1) we arrive at (4.2). □

As a corollary, taking $y = \lambda_1(\frac{1}{\nu})^z$, for any positive real numbers z and $\lambda_1 \leqslant \lambda_2 \leqslant +\infty$ and any $\nu \in \Upsilon''_k$ we have

$$\sum_{\lambda_1(\frac{1}{\nu})^z \leqslant |\boldsymbol{s}| \leqslant \lambda_2(\frac{1}{\nu})^z} \hat{a}^2_{\boldsymbol{s}}(\nu) \leqslant \lambda_1^{-2k} c''_k \nu^{2kz - 2p''(k)}.$$

Lemma 5. *Suppose that there exist real numbers $0 \leqslant k_1 < k < k_2$, $p''(k_1)$, $p'(k)$, $p''(k_2)$, $c''_{k_1} > 0$, $c'_k > 0$, $c''_{k_2} > 0$ and sets $\Upsilon''_{k_1}, \Upsilon'_k, \Upsilon''_{k_2} \subset (0, 1]$ such that*

$$\sum_{\boldsymbol{s}\in\mathbb{Z}^n} |\boldsymbol{s}|^{2k_i} \hat{a}^2_{\boldsymbol{s}}(\nu) \leqslant c''_{k_i} \left(\tfrac{1}{\nu}\right)^{2p''(k_i)} \quad \text{for any } \nu \in \Upsilon''_{k_i}, \quad i = 1, 2 \tag{4.3}$$

and

$$\sum_{\boldsymbol{s}\in\mathbb{Z}^n} |\boldsymbol{s}|^{2k} \hat{a}^2_{\boldsymbol{s}}(\nu) \geqslant c'_k \left(\tfrac{1}{\nu}\right)^{2p'(k)} \quad \text{for any } \nu \in \Upsilon'_k. \tag{4.4}$$

Then for any $\mu \in (0, 1)$ and any real $A \leqslant \bar{A}(\nu)$ and $B \geqslant \bar{B}(\nu)$ where

$$\bar{A}(\nu) = \left(\tfrac{\mu}{2}\tfrac{c'_k}{c''_{k_1}}\right)^{\frac{1}{2k-2k_1}} \left(\tfrac{1}{\nu}\right)^{\frac{p'(k)-p''(k_1)}{k-k_1}} \quad \text{and} \quad \bar{B}(\nu) = \left(\tfrac{2}{\mu}\tfrac{c''_{k_2}}{c'_k}\right)^{\frac{1}{2k_2-2k}} \left(\tfrac{1}{\nu}\right)^{\frac{p''(k_2)-p'(k)}{k_2-k}} \tag{4.5}$$

and any $\nu \in \Upsilon''_{k_1} \cap \Upsilon'_k \cap \Upsilon''_{k_2}$ *we have*

$$\sum_{A<|\boldsymbol{s}|<B} |\boldsymbol{s}|^{2k}\hat{a}_{\boldsymbol{s}}^2 \geqslant (1-\mu) \sum_{\boldsymbol{s}\in\mathbb{Z}^n} |\boldsymbol{s}|^{2k}\hat{a}_{\boldsymbol{s}}^2. \tag{4.6}$$

Proof. For any real $A > 0$ and $B > 0$ we have

$$\sum_{|\boldsymbol{s}|\leqslant A} |\boldsymbol{s}|^{2k}\hat{a}_{\boldsymbol{s}}^2 = \sum_{|\boldsymbol{s}|\leqslant A} |\boldsymbol{s}|^{2k-2k_1}|\boldsymbol{s}|^{2k_1}\hat{a}_{\boldsymbol{s}}^2 \leqslant A^{2k-2k_1} \sum_{|\boldsymbol{s}|\leqslant A} |\boldsymbol{s}|^{2k_1}\hat{a}_{\boldsymbol{s}}^2 \leqslant A^{2k-2k_1} \sum_{\boldsymbol{s}\in\mathbb{Z}} |\boldsymbol{s}|^{2k_1}\hat{a}_{\boldsymbol{s}}^2$$

and

$$\sum_{|\boldsymbol{s}|\geqslant B} |\boldsymbol{s}|^{2k}\hat{a}_{\boldsymbol{s}}^2 = \sum_{|\boldsymbol{s}|\geqslant B} |\boldsymbol{s}|^{2k-2k_2}|\boldsymbol{s}|^{2k_2}\hat{a}_{\boldsymbol{s}}^2 \leqslant B^{2k-2k_2} \sum_{|\boldsymbol{s}|\geqslant B} |\boldsymbol{s}|^{2k_2}\hat{a}_{\boldsymbol{s}}^2 \leqslant B^{2k-2k_2} \sum_{\boldsymbol{s}\in\mathbb{Z}} |\boldsymbol{s}|^{2k_2}\hat{a}_{\boldsymbol{s}}^2.$$

Under the condition $A \leqslant \bar{A}(\nu)$ for any $\nu \in \Upsilon''_{k_1} \cap \Upsilon'_k$ we get

$$\sum_{|\boldsymbol{s}|\leqslant A} |\boldsymbol{s}|^{2k}\hat{a}_{\boldsymbol{s}}^2 \leqslant \tfrac{\mu}{2} c'_k \left(\tfrac{1}{\nu}\right)^{2p'(k)} \leqslant \tfrac{\mu}{2} \sum_{\boldsymbol{s}\in\mathbb{Z}} |\boldsymbol{s}|^{2k}\hat{a}_{\boldsymbol{s}}^2.$$

Under the condition $B \geqslant \bar{B}(\nu)$ for any $\nu \in \Upsilon'_k \cap \Upsilon''_{k_2}$ we get

$$\sum_{|\boldsymbol{s}|\geqslant B} |\boldsymbol{s}|^{2k}\hat{a}_{\boldsymbol{s}}^2 \leqslant \tfrac{\mu}{2} c'_k \left(\tfrac{1}{\nu}\right)^{2p'(k)} \leqslant \tfrac{\mu}{2} \sum_{\boldsymbol{s}\in\mathbb{Z}} |\boldsymbol{s}|^{2k}\hat{a}_{\boldsymbol{s}}^2$$

and (4.6) follows. □

Due to the inequality

$$S_k \leqslant S_{k_1}^{\frac{k_2-k}{k_2-k_1}} S_{k_2}^{\frac{k-k_1}{k_2-k_1}}, \tag{4.7}$$

where $S_k = S_k(\nu) = \sum_{\boldsymbol{s}\in\mathbb{Z}^n} |\boldsymbol{s}|^{2k}\hat{a}_{\boldsymbol{s}}^2(\nu)$, it follows that $\bar{A} < \bar{B}$. If the closure of the set $\Upsilon''_{k_1} \cap \Upsilon'_k \cap \Upsilon''_{k_2}$ contains zero, then the powers p'' and p' in (4.3) and (4.4) satisfy the following convexity property

$$\frac{p'(k) - p''(k_1)}{k - k_1} \leqslant \frac{p''(k_2) - p'(k)}{k_2 - k} \tag{4.8}$$

or, equivalently,

$$p'(k) \leqslant \frac{(k-k_1)p''(k_2) + (k_2-k)p''(k_1)}{k_2 - k_1}. \tag{4.9}$$

The above two lemmas will be used to obtain bounds for the distribution of the Fourier coefficients of a function $\boldsymbol{u}$ such that $\|\boldsymbol{u}\|_k^2 \in [c'_k\nu^{-2p'(k)}, c''_k\nu^{-2p''(k)}]$ for each k.

We believe that for solutions of many types of PDE's and, in particular, for solutions of (1.1) we have $p'(k) = p''(k)$ and in (4.8) and (4.9) we have equalities. According to numerics of D. Jefferson (see [6]) this is the case for the complex Ginzburg-Landau equation. Moreover, for 1D Burgers-type equations this result is proven analytically (see [1]). In the case of equality in (4.8) and (4.9), lemma 5 allows to write the lower estimate for the narrowest (in terms of powers of viscosity) layer of the wave-numbers. Moreover, there is an upper bound for the sum over the same layer which coincides (in terms of powers of the viscosity) with the lower bounds. Indeed, using (4.7) we obtain

$$\sum_{\bar{A}<|\boldsymbol{s}|<\bar{B}} |\boldsymbol{s}|^{2k}\hat{a}_{\boldsymbol{s}}^2 \leqslant \sum_{\boldsymbol{s}\in\mathbb{Z}^n} |\boldsymbol{s}|^{2k}\hat{a}_{\boldsymbol{s}}^2 \leqslant (c''_{k_1})^{\frac{k_2-k}{k_2-k_1}} (c''_{k_2})^{\frac{k-k_1}{k_2-k_1}} \left(\tfrac{1}{\nu}\right)^{2p'(k)},$$

while by lemma 5 we have

$$\sum_{\bar{A}<|\boldsymbol{s}|<\bar{B}} |\boldsymbol{s}|^{2k}\hat{a}_{\boldsymbol{s}}^2 \geqslant (1-\mu)c'_k \left(\tfrac{1}{\nu}\right)^{2p'(k)}.$$

Under assumptions of lemma 5 we cannot, in general, expect any lower bound outside the layer $\bar{A} < |\boldsymbol{s}| < \bar{B}$. Indeed, let $\gamma > 0$ and b be any real numbers and suppose that $S_k(\nu) \leqslant c''_k\left(\frac{1}{\nu}\right)^{2k\gamma-2b}$ and $S_k(\nu) \geqslant c'_k\left(\frac{1}{\nu}\right)^{2k\gamma-2b}$. Then the coefficients $\hat{a}^2_{\boldsymbol{s}}(\nu)$ could be as follows:

$$\hat{a}^2_{\boldsymbol{s}}(\nu) = \begin{cases} \nu^{2b} & \text{for } \boldsymbol{s} = ([\frac{1}{\nu^\gamma}], 0, \dots, 0), \\ 0 & \text{otherwise.} \end{cases}$$

To connect the results obtained with the turbulence theory we give a *possible* rigorous definition of K-O type law and then obtain bounds for two the most important parameters of the law.

Definition 4. *We say that positive quantities* $\hat{a}^2_{\boldsymbol{s}}(\nu)$ *obey a* K-O type spectral law *if there exist positive real numbers* ν_0, $\kappa_1 \leqslant \kappa_2$, $\varkappa$, c, C, C_1 *positive real functions* $\sigma_1(\nu)$ *and* $\sigma_2(\nu)$ *such that* $\log(\sigma_i(\nu)) = \bar{\bar{o}}(\log(\nu^{-1}))$ *as* $\nu \to 0$ *(and* $\sup \frac{\sigma_1(\nu)}{\sigma_2(\nu)} < 1$ *if* $\kappa_1 = \kappa_2$*) such that for any* $\nu \in (0, \nu_0)$ *we have*

1. $\sum_{r-C_1<|\boldsymbol{s}|<r+C_1} \hat{a}^2_{\boldsymbol{s}}(\nu) \leqslant C\left(\frac{1}{r}\right)^{\varkappa}$ *for* $r > \sigma_1(\nu)\left(\frac{1}{\nu}\right)^{\kappa_1}$
2. $\sum_{r-C_1<|\boldsymbol{s}|<r+C_1} \hat{a}^2_{\boldsymbol{s}}(\nu) \geqslant c\left(\frac{1}{r}\right)^{\varkappa}$ *for* $\sigma_1(\nu)\left(\frac{1}{\nu}\right)^{\kappa_1} < r < \sigma_2(\nu)\left(\frac{1}{\nu}\right)^{\kappa_2}$
3. *There exist function* $\varphi(k) = \bar{\bar{o}}(k)$ *as* $k \to \infty$ *and positive functions* $C(k)$, $\nu(k)$ *such that for any sufficiently large* k, *any* $r > 0$, *and any* $\nu \in (0, \nu(k))$ *we have*

$$\sum_{|\boldsymbol{s}|\geqslant r} \hat{a}^2_{\boldsymbol{s}}(\nu) \leqslant C(k)\left(r\nu^{\kappa_2}\right)^{-k} \nu^{-\varphi(k)}. \tag{4.10}$$

We will interpret the quantities $\hat{a}^2_{\boldsymbol{s}}(\nu)$ as the Fourier coefficients of a family of functions depending on the parameter ν. Then the first and the second conditions means that the L_2-norm that is carried by the modes on the sphere of radius r, up to constants, behaves as $r^{-\varkappa}$ for wave numbers r from the inertial range $r_1(\nu) < r < r_2(\nu)$. While for $r \geqslant r_2(\nu)$ we have an upper bound only. The third condition, in particular, means that for fixed ν the sum $\sum_{|\boldsymbol{s}|\geqslant r} \hat{a}^2_{\boldsymbol{s}}(\nu)$ becomes small for $r > \nu^{-\kappa_2}$ and decays faster than any finite power of $\frac{1}{r}$. Also it says that for any $\varepsilon > 0$ and $M > 0$ the quantity $\sum_{|\boldsymbol{s}|\geqslant\nu^{-\kappa_2-\varepsilon}} \hat{a}^2_{\boldsymbol{s}}(\nu)$ decays faster than ν^M as $\nu \to 0$.

Below we give a simple sufficient condition which implies (4.10). This condition covers the case of linear dependence of the power of the viscosity in the upper bounds for Sobolev norm (see (4.1)) on the its number.

Proposition 5. *Suppose that for any* $k > 0$ *and any* $\nu \in (0,1)$ *we have* $\sum |\boldsymbol{s}|^{2k}\hat{a}^2_{\boldsymbol{s}}(\nu) \leqslant c''_k\nu^{-2\kappa_2 k+q}$. *Then (4.10) holds with* $C(k) = c''_{k/2}$, $\varphi(k) = q$ *and* $\nu(k) = 1$.

Proof follows from lemma 4. □

Lemma 6. *Suppose that the quantities* $\hat{a}^2_{\boldsymbol{s}}(\nu)$ *obey a K-O type spectral law in the sense of definition 4. Then the values* $\varkappa$ *and* κ_2 *are uniquely defined.*

Proof. The statement about $\varkappa$ is obvious since the interval $(\nu^{-\kappa_1}\sigma_1(\nu), \nu^{-\kappa_2}\sigma_2(\nu))$ is non-empty for small ν. The second statement follows from the relation:

$$\kappa_2 = \inf\{\kappa : \forall M \text{ the sum } \sum_{|\boldsymbol{s}|>\nu^{-\kappa}} \hat{a}^2_{\boldsymbol{s}}(\nu) \text{ decays faster than } \nu^M \text{ as } \nu \to 0\}. \tag{4.11}$$

□

The number $\varkappa$ is called the exponent of the K-O law and the number κ_2 is the power of the Kolmogorov dissipative scale. The quantity ν^{κ_2} is the Kolmogorov dissipative scale.

The value κ_1 (as well as c, C, C_1 and ν_0) is not, in general, uniquely defined. For example if $\kappa_1 < \kappa_2$ then we can replace κ_1 with any real number in $(\kappa_1, \kappa_2]$.

Lemma 7. *Let us assume that we are given:*

i) real numbers p, z and $0 < h_1 \leqslant h_2$;

ii) positive real functions $\sigma_1(\nu)$ and $\sigma_2(\nu)$ such that $\log(\sigma_i(\nu)) = \bar{o}(\log(\nu^{-1}))$ as $\nu \to 0$ (and $\sup \frac{\sigma_1(\nu)}{\sigma_2(\nu)} < 1$ if $h_1 = h_2$).

Suppose that the inequalities

$$a) \quad \int_{\sigma_1(\nu)\,\nu^{-h_1}}^{\sigma_2(\nu)\,\nu^{-h_2}} x^z dx \leqslant C''\nu^{-p} \qquad\qquad b) \quad \int_{\sigma_1(\nu)\,\nu^{-h_1}}^{\sigma_2(\nu)\,\nu^{-h_2}} x^z dx \geqslant C'\nu^{-p}$$

holds with some positive constants C'' and C' for a set Υ of values ν that contains 0 in its closure. Then

$$a) \quad \begin{array}{ll} z \leqslant \frac{p}{h_2} - 1 & \text{for } p > 0, \\ z \leqslant -1 & \text{for } p = 0, \\ z < -1 & \text{for } p = 0 \text{ and } h_1 < h_2, \\ z \leqslant \frac{p}{h_1} - 1 & \text{for } p < 0, \end{array} \qquad\qquad b) \quad \begin{array}{ll} z \geqslant \frac{p}{h_2} - 1 & \text{for } p > 0, \\ z \geqslant -1 & \text{for } p = 0, \\ z \geqslant \frac{p}{h_1} - 1 & \text{for } p < 0. \end{array}$$

Proof. For the brevity, we write σ_1 and σ_2 for $\sigma_1(\nu)$ and $\sigma_2(\nu)$. It is clear that for small enough ν (such that $1 < \sigma_1\nu^{-h_1} < \sigma_2\nu^{-h_2}$) the integral $\int_{\sigma_1\nu^{-h_1}}^{\sigma_2\nu^{-h_2}} x^z dx$ increases with z. The rest of the proof follows from the following direct calculations:

$$\int_{\sigma_1\nu^{-h_1}}^{\sigma_2\nu^{-h_2}} x^z dx = \begin{cases} \frac{1}{z+1}\sigma_2^{z+1}\nu^{-h_2(z+1)}\left(1 - \left(\frac{\sigma_1}{\sigma_2}\right)^{z+1}\nu^{(h_2-h_1)(z+1)}\right) & \text{if } z > -1, \\ (h_2 - h_1)\log(\nu^{-1}) + \log(\sigma_2/\sigma_1) & \text{if } z = -1, \\ \frac{-1}{z+1}\sigma_1^{z+1}\nu^{-h_1(z+1)}\left(1 - \left(\frac{\sigma_2}{\sigma_1}\right)^{z+1}\nu^{(h_1-h_2)(z+1)}\right) & \text{if } z < -1. \end{cases}$$

□

Lemma 8. a) *Under the assumptions of lemma 4 suppose that $\overline{\Upsilon_k''} \ni 0$ and that the quantities $\hat{a}_{\boldsymbol{s}}^2(\nu)$ obey a K-O type spectral law. Then*

$$\begin{array}{ll} \varkappa \geqslant -\frac{2p''(k)}{\kappa_2} + 2k + 1 & \text{if } p''(k) > 0, \\ \varkappa \geqslant 1 + 2k & \text{if } p''(k) = 0, \\ \varkappa > 1 + 2k & \text{if } p''(k) = 0 \text{ and } \kappa_1 < \kappa_2, \\ \varkappa \geqslant -\frac{2p''(k)}{\kappa_1} + 2k + 1 & \text{if } p''(k) < 0. \end{array}$$

b) *Suppose that there are real sequences $\{k_i\}$ and $\{\nu(k_i)\}$, $k_i \to \infty$, such that lemma 4 holds for any $k = k_i$ with $\Upsilon_{k_i}'' = (0, \nu(k_i))$. Then*

$$\kappa_2 \leqslant \liminf \frac{p''(k_i)}{k_i}. \tag{4.12}$$

Proof. a) By condition 2 of the definition 4 we have

$$\sum_{r-C_1<|\boldsymbol{s}|<r+C_1} |\boldsymbol{s}|^{2k}\hat{a}_{\boldsymbol{s}}^2(\nu) \geqslant const\, r^{2k-\varkappa} \quad \text{for} \quad \sigma_1(\nu)\left(\tfrac{1}{\nu}\right)^{\kappa_1} < r < \sigma_2(\nu)\left(\tfrac{1}{\nu}\right)^{\kappa_2}.$$

Hence (with a different constant) we have

$$\sum_{\sigma_1(\nu)\,\nu^{-\kappa_1}<|\boldsymbol{s}|<\sigma_1(\nu)\,\nu^{-\kappa_1}} |\boldsymbol{s}|^{2k}\hat{a}_{\boldsymbol{s}}^2(\nu) \geqslant const \int_{\sigma_1\nu^{-\kappa_1}}^{\sigma_2\nu^{-\kappa_2}} x^{2k-\varkappa} dx.$$

Using assumption (4.1) we obtain the following inequality (again with a different constant)

$$\int_{\sigma_1\nu^{-\kappa_1}}^{\sigma_2\nu^{-\kappa_2}} x^{2k-\varkappa} dx \leqslant const\, \nu^{-2p''(k)}.$$

Now we apply lemma 7.a with $z = 2k - \varkappa$, $p = 2p''(k)$, $h_1 = \kappa_1$, $h_2 = \kappa_2$, and $\Upsilon = \Upsilon''_k \cap (0, \nu_0)$ to complete the proof of the first part of the lemma.

b) We prove that for any $\gamma \geqslant \liminf \frac{p''(k_i)}{k_i}$ condition 3 of the definition 4 holds if the value κ_2 in (4.10) is replaced with γ.

To do this we take a subsequence $\{k_{j_i}\}$ such that $\liminf \frac{p''(k_i)}{k_i} = \lim \frac{p''(k_{j_i})}{k_{j_i}}$. Then

$$\max\{0, \frac{p''(k_{j_i})}{k_{j_i}} - \gamma\} = \bar{\bar{o}}(1) \text{ as } i \to \infty. \tag{4.13}$$

Now applying (4.2) for each $k = k_{j_i}$ we obtain

$$\sum_{|\boldsymbol{s}| \geqslant r} \hat{a}^2_{\boldsymbol{s}}(\nu) \leqslant C(2k_{j_i})\,(r\nu^{\gamma})^{-2k_{j_i}}\,\nu^{-\varphi(2k_{j_i})} \quad \text{for any } i \text{ and any } \nu \in (0, \nu(k_{j_i})). \tag{4.14}$$

where $C(2k_{j_i}) = c''_{k_{j_i}}$ and $\varphi(2k_{j_i}) = \max\{0, 2p''(k_{j_i}) - 2k_{j_i}\gamma\}$ $(= \bar{\bar{o}}(2k_{j_i})$ due to (4.13)). Using the interpolation we see that (4.14) holds when $2k_{j_i}$ is replaced with any $k \geqslant 0$ (and the functions $C(\cdot)$ and $\varphi(\cdot)$ are determined according to this interpolation). It follows that $\kappa_2 \leqslant \gamma$. Hence $\kappa_2 \leqslant \liminf \frac{p''(k_i)}{k_i}$. □

Lemma 9. a) *Under the assumptions of lemma 5 suppose that* $\overline{\Upsilon''_{k_1} \cap \Upsilon'_k \cap \Upsilon''_{k_2}} \ni 0$, *and that the quantities* $\hat{a}^2_{\boldsymbol{s}}(\nu)$ *obey a K-O type spectral law. Then*

$$\kappa_2 \geqslant \frac{p'(k) - p''(k_1)}{k - k_1}. \tag{4.15}$$

b) *If, in addition, we have* $\kappa_1 < \frac{p'(k)-p''(k_1)}{k-k_1}$ *then*

$$\begin{aligned} \varkappa &\leqslant -\frac{2p'(k)}{\min\left\{\kappa_2, \frac{p''(k_2)-p'(k)}{k_2-k}\right\}} + 2k + 1 && \text{if } p'(k) > 0,\\ \varkappa &\leqslant 1 + 2k && \text{if } p'(k) = 0,\\ \varkappa &\leqslant -2p'(k)\frac{k-k_1}{p'(k)-p''(k_1)} + 2k + 1 && \text{if } p'(k) < 0. \end{aligned}$$

Proof. a) We denote $p_- = \frac{p'(k)-p''(k_1)}{k-k_1}$ and $p_+ = \frac{p''(k_2)-p'(k)}{k_2-k}$. To prove (4.15) we show that for any $\varepsilon > 0$ the sum $\sum_{|\boldsymbol{s}|>\nu^{-(p_- - \varepsilon)}} \hat{a}^2_{\boldsymbol{s}}(\nu)$ decays as $\nu \to 0$ not faster than some finite power of ν.

Fix any $\varepsilon > 0$. Then for small enough ν such that $\nu^{-(p_- - \varepsilon)} < \bar{A}(\nu)$ (the quantities $\bar{A}(\nu)$ and $\bar{B}(\nu)$ are defined by (4.5) with $\mu = 1/2$) due to lemma 5 we have

$$\sum_{|\boldsymbol{s}|>\nu^{-(p_- - \varepsilon)}} \hat{a}^2_{\boldsymbol{s}}(\nu) \geqslant \sum_{\bar{A}(\nu)<|\boldsymbol{s}|<\bar{B}(\nu)} \hat{a}^2_{\boldsymbol{s}}(\nu) \geqslant (\bar{B}(\nu))^{-2k} \sum_{\bar{A}(\nu)<|\boldsymbol{s}|<\bar{B}(\nu)} |\boldsymbol{s}|^{2k} \hat{a}^2_{\boldsymbol{s}}(\nu) \geqslant const\, \nu^{2kp_+ - 2p'(k)}.$$

b) Let $\bar{\bar{B}}_\varepsilon(\nu) = \min\{\bar{B}(\nu), \nu^{-(\kappa_2+\varepsilon)}\}$. First we claim that for small enough ν we have

$$\sum_{\bar{A}(\nu)<|\boldsymbol{s}|<\bar{\bar{B}}_\varepsilon(\nu)} |\boldsymbol{s}|^{2k} \hat{a}^2_{\boldsymbol{s}} \geqslant const\, \nu^{-2p'(k)}.$$

Indeed, if $\bar{B}(\nu) \leqslant \nu^{-(\kappa_2+\varepsilon)}$, then the inequality directly follows from lemma 5. If $\bar{B}(\nu) > \nu^{-(\kappa_2+\varepsilon)}$, then it follows from lemma 5 and the fact that the sum $\sum_{|\boldsymbol{s}|>\nu^{-(\kappa_2+\varepsilon)}} \hat{a}^2_{\boldsymbol{s}}(\nu)$ decays faster then any power of ν.

The assumption $\kappa_1 < p_-$ implies that for small enough ν we have $\sigma_1(\nu)\nu^{-\kappa_1} < \bar{A}(\nu)$ and so, by the condition 1 of definition 4, we have

$$\sum_{\bar{A}(\nu)<|\boldsymbol{s}|<\bar{\bar{B}}_\varepsilon(\nu)} |\boldsymbol{s}|^{2k}\hat{a}_{\boldsymbol{s}}^2 \leqslant const \int_{\bar{A}(\nu)}^{\bar{\bar{B}}_\varepsilon(\nu)} x^{2k-\varkappa}dx.$$

This inequality and the claim above imply

$$\int_{\bar{A}(\nu)}^{\bar{\bar{B}}_\varepsilon(\nu)} x^{2k-\varkappa}dx \geqslant const\, \nu^{-2p'(k)}.$$

Now we apply lemma 7.b and take the limit $\varepsilon \to 0$ to complete the proof of the second part of the lemma. □

4.2 Back to the Burgers equation

Here we apply the results of the previous subsection to estimate distribution of the averaged Fourier coefficients for solutions of (1.1).

For the sequel we note that if

$$q_n = \left[\tfrac{n}{2}\right] + 1 \tag{4.16}$$

then for any vector-valued function $\boldsymbol{u}$ on the torus $\mathbb{T}^n$, any $k \geqslant q_n + 1$ and any index j we have

$$\|\boldsymbol{u}\|_k^2 \geqslant K_n^2 \Big|\tfrac{\partial^{k-q_n}\boldsymbol{u}}{\partial x_j^{k-q_n}}\Big|_{L_\infty}^2. \tag{4.17}$$

where

$$K_n = \ell^{n/2}\left(\tfrac{2\pi}{\ell}\right)^{2q_n}\Big(\sum_{\boldsymbol{s}\in\mathbb{Z}^n\setminus\{0\}} \tfrac{1}{|\boldsymbol{s}|^{2q_n}}\Big)^{-\frac{1}{2}}. \tag{4.18}$$

Lemma 10. *Let $\boldsymbol{u} = \boldsymbol{u}^\nu(t,\boldsymbol{x})$ be a solution of (1.1) with an initial state $\boldsymbol{u}_0$.*

i) Let ν_0 and T be any positive numbers. Then for each $k \geqslant 0$ there exists $\widetilde{C}_k = \widetilde{C}_k(T,\nu_0)$ such that for any positive $\nu < \nu_0$ we have

$$\frac{1}{T}\int_0^T \|\boldsymbol{u}\|_k^2 dt \leqslant \widetilde{C}_k \left(\tfrac{1}{\nu}\right)^{2k}. \tag{4.19}$$

ii) Suppose that $\boldsymbol{u}_0$ is non-degenerate. Let $T = T(\boldsymbol{f},\boldsymbol{u}_0)$ be given by (3.16). Suppose that the forcing term $\boldsymbol{h}$ in (1.1) satisfies the condition $H(T) < \frac{c}{2}$, where c is given by (3.33). Then for any integer $k \geqslant 2 + q_n$ and for any $\nu > 0$, we have

$$\frac{1}{T}\int_0^T \|\boldsymbol{u}\|_k^2 dt \geqslant K_n^2 r_{k-q_n}^2 \left(\tfrac{1}{\nu}\right)^{k-q_n}. \tag{4.20}$$

Here r_k, q_n and K_n is given by (3.11), (4.16) and (4.18), respectively.

Proof. i) Using (1.6) we arrive at (4.19) with $\widetilde{C}_k = R_k^2(T)\max\{1, \frac{\|\boldsymbol{u}_0\|_k^2}{R_k(0)^2}\nu_0^{2k}, \frac{\sup_{[0,t]}\|\boldsymbol{h}\|_{k-1}^2}{R_k(0)^2}\nu_0^{2k}\}$.

ii) Using inequality (4.17), the Cauchy-Schwarz inequality and theorem 2 we obtain:

$$\frac{1}{T}\int_0^T \|\boldsymbol{u}\|_k^2 dt \geqslant K_n^2 \frac{1}{T}\int_0^T \Big|\tfrac{\partial^{k-q_n}\boldsymbol{u}}{\partial x_j^{k-q_n}}\Big|_{L_\infty}^2 dt \geqslant K_n^2 \left(\frac{1}{T}\int_0^T \Big|\tfrac{\partial^{k-q_n}\boldsymbol{u}}{\partial x_j^{k-q_n}}\Big|_{L_\infty} dt\right)^2 \geqslant K_n^2 \frac{r_{k-q_n}^2}{\nu^{k-q_n}}.$$ □

Consider the orthonormal basis on $L_2(\mathbb{T}^n)$, formed by the exponents $e_{\boldsymbol{s}}(\cdot)$, $\boldsymbol{s} \in \mathbb{Z}^n$, where

$$e_{\boldsymbol{s}}(\boldsymbol{x}) = \tfrac{1}{\ell^{n/2}}\exp\left(\tfrac{2\pi i\boldsymbol{s}\boldsymbol{x}}{\ell}\right).$$

The Fourier expansion of $\boldsymbol{u}(\boldsymbol{x})$ has the form

$$\boldsymbol{u}(t,\boldsymbol{x}) = \sum_{\boldsymbol{s}\in\mathbb{Z}^n} \hat{\boldsymbol{u}}_{\boldsymbol{s}}(t) e_{\boldsymbol{s}}(\boldsymbol{x}), \quad \text{where} \quad \hat{\boldsymbol{u}}_{\boldsymbol{s}}(t) = \int_{\mathbb{T}^n} \boldsymbol{u}(t,\boldsymbol{x})\overline{e_{\boldsymbol{s}}(\boldsymbol{x})} d\boldsymbol{x}.$$

For the Sobolev quasinorm $\|\cdot\|_k$ (1.5) we have $\|\boldsymbol{u}\|_k^2 = \left(\frac{2\pi}{\ell}\right)^{2k} \sum_{\boldsymbol{s}\in\mathbb{Z}^n} |\boldsymbol{s}|^{2k} |\hat{\boldsymbol{u}}_{\boldsymbol{s}}|^2$.

Suppose that the initial state $\boldsymbol{u}_0$ is a non-degenerate vector field. Consider the corresponding solution of equation (1.1) and define the quantities $\hat{a}_{\boldsymbol{s}}^2(\nu)$ by the formula:

$$\hat{a}_{\boldsymbol{s}}^2 = \frac{1}{T}\int_0^T |\hat{\boldsymbol{u}}_{\boldsymbol{s}}(t)|^2 dt, \tag{4.21}$$

where $T = T(\boldsymbol{f}, \boldsymbol{u}_0)$ is defined by (3.16). Then we have

$$\sum_{\boldsymbol{s}\in\mathbb{Z}^n} |\boldsymbol{s}|^{2k} \hat{a}_{\boldsymbol{s}}^2(\nu) = \left(\frac{\ell}{2\pi}\right)^{2k} \frac{1}{T}\int_0^T \|\boldsymbol{u}^\nu\|_k^2 \, dt.$$

Lemma 10 implies that for any k and any $\nu \in (0,\nu_0)$ we have the following inequality

$$\sum_{\boldsymbol{s}\in\mathbb{Z}^n} |\boldsymbol{s}|^{2k} \hat{a}_{\boldsymbol{s}}^2(\nu) \leqslant \left(\tfrac{\ell}{2\pi}\right)^{2k} \widetilde{C}_k \left(\tfrac{1}{\nu}\right)^{2k}. \tag{4.22}$$

Under the assumptions of the second part of lemma 10 we see that for any $\nu > 0$ and any integer $k \geqslant q_n + 2$ there is the inequality:

$$\sum_{\boldsymbol{s}\in\mathbb{Z}^n} |\boldsymbol{s}|^{2k} \hat{a}_{\boldsymbol{s}}^2(\nu) \geqslant K_n^2 r_{k-q_n}^2 \left(\tfrac{\ell}{2\pi}\right)^{2k} \left(\tfrac{1}{\nu}\right)^{k-q_n}. \tag{4.23}$$

We recall that $q_n = \left[\frac{n}{2}\right] + 1$ (4.16).

Theorem 6. *Let $\boldsymbol{u} = \boldsymbol{u}^\nu(t,\boldsymbol{x})$ be the solution of (1.1) with a non-degenerate initial state $\boldsymbol{u}_0$. Take $T = T(\boldsymbol{f}, \boldsymbol{u}_0)$ by (3.16) and consider the averaged Fourier coefficients $\hat{a}_{\boldsymbol{s}}^2(\nu)$ given by (4.21). Suppose that the forcing term $\boldsymbol{h}$ in (1.1) satisfies the condition $H(T) < \frac{c}{2}$, where c is given by the right hand side of (3.33). Then for any $k \geqslant 2 + q_n$, where q_n is given by (4.16), there exists c_k such that for any $\varepsilon > 0$ and any small enough ν we have*

$$\sum_{\left(\frac{1}{\nu}\right)^{\frac{1}{2} - \frac{q_n}{2k} - \varepsilon} < |\boldsymbol{s}| < \left(\frac{1}{\nu}\right)^{1+\varepsilon}} \hat{a}_{\boldsymbol{s}}^2(\nu) \geqslant const_k \nu^{k+2k\varepsilon+q_n}. \tag{4.24}$$

For any $k \geqslant 0$ and $\nu_0 > 0$ there exists $C_k = C_k(\nu_0)$ such that for any positive real numbers z and $\lambda_1 < \lambda_2$, and any $\nu < \nu_0$ we have

$$\sum_{\lambda_1\left(\frac{1}{\nu}\right)^z \leqslant |\boldsymbol{s}| \leqslant \lambda_2\left(\frac{1}{\nu}\right)^z} \hat{a}_{\boldsymbol{s}}^2(\nu) \leqslant \lambda_1^{-2k} C_k \, \nu^{2k(z-1)}. \tag{4.25}$$

Besides, for any $y > 0$ we have

$$\sum_{|\boldsymbol{s}|\geqslant y} \hat{a}_{\boldsymbol{s}}^2(\nu) \leqslant C_k (y\nu)^{-2k}. \tag{4.26}$$

Proof. The quantities $\hat{a}_s^2(\nu)$ satisfy inequalities (4.23) and (4.22). Hence we can apply lemmas 4 and 5 with $p''(k) = k$, $p'(k) = k/2 - q_n/2$ for $k \geqslant q_n + 2$, $c_k'' = \left(\frac{\ell}{2\pi}\right)^{2k} \widetilde{C}_k$ $c_k' = \left(\frac{\ell}{2\pi}\right)^{2k} K_n^2 r_{k-q_n}^2$. Choose $k_1 = 0$ and $k_2 > k$ such that

$$\frac{k_2 - (k/2 - q_n/2)}{k_2 - k} < 1 + \varepsilon,$$

and $\nu(k,\varepsilon,c_i',c_i'')$ such that for any positive $\nu < \nu(k,\varepsilon,c_i',c_i'')$ we have

$$\left(\tfrac{1}{\nu}\right)^{\frac{1}{2}-\frac{q_n}{2k}-\varepsilon} < \bar{A}(\nu) \quad \text{and } \bar{B}(\nu) > \left(\tfrac{1}{\nu}\right)^{1+\varepsilon},$$

where $\bar{A}(\nu)$ and $\bar{B}(\nu)$ are defined by (4.5) with $\mu = 1/2$. Now we apply lemma 5 to obtain

$$\sum_{\left(\frac{1}{\nu}\right)^{\frac{1}{2}-\frac{q_n}{2k}-\varepsilon}<|\boldsymbol{s}|<\left(\frac{1}{\nu}\right)^{1+\varepsilon}} |\boldsymbol{s}|^{2k}\hat{a}_{\boldsymbol{s}}^2 \geqslant \tfrac{1}{2}c_k'\left(\tfrac{1}{\nu}\right)^{k-q_n}.$$

Using the inequality

$$\sum_{\left(\frac{1}{\nu}\right)^{\frac{1}{2}-\frac{q_n}{2k}-\varepsilon}<|\boldsymbol{s}|<\left(\frac{1}{\nu}\right)^{1+\varepsilon}} \hat{a}_{\boldsymbol{s}}^2 \geqslant \nu^{2k(1+\varepsilon)} \sum_{\left(\frac{1}{\nu}\right)^{\frac{1}{2}-\frac{q_n}{2k}-\varepsilon}<|\boldsymbol{s}|<\left(\frac{1}{\nu}\right)^{1+\varepsilon}} |\boldsymbol{s}|^{2k}\hat{a}_{\boldsymbol{s}}^2$$

we arrive at (4.24) with $const_k = c_k'/2$.

Using inequality (4.22) and lemma 4 we arrive at (4.25) and (4.26) with $C_k(\nu_0) = c_k''$. □

Under the assumptions of theorem 6, we have the following conditional result:

Theorem 7. *If the quantities $\hat{a}_{\boldsymbol{s}}^2(\nu)$, defined by (4.21), obey a K-O type spectral law, then the power of the Kolmogorov dissipation scale $\kappa_2 \in [1/2, 1]$. For the power of the spectral law $\varkappa$ we have $\varkappa \geqslant 1$, and $\varkappa > 1$ if $\kappa_1 < \kappa_2$.*

Suppose, in addition, that the energy range is small enough, that is $\kappa_1 < \frac{1}{2} - \frac{q_n}{2k}$ for some integer $k \geqslant q_n + 2$; then we have $\varkappa \leqslant k + q_n + 1$. In particular, if $\kappa_1 < \frac{1}{q_n+2}$ then we have $\varkappa \leqslant 2q_n + 3$.

Proof. For any $k \geqslant 0$ the quantities $\hat{a}_s^2(\nu)$ satisfy inequalities (4.1) with $p''(k) = k$, while for $k \geqslant q_n + 2$ the quantities $\hat{a}_s^2(\nu)$ satisfy inequalities (4.4) with $p'(k) = \frac{1}{2}(k - q_n)$. By (4.12) we have $\kappa_2 \leqslant 1$. Taking any k_1 in (4.15) and then passing $k \to \infty$ we obtain $\kappa_2 \geqslant 1/2$. Using lemma 8.a with $k = 0$ we obtain that $\varkappa \geqslant 1$ and $\varkappa > 1$ if $\kappa_1 < \kappa_2$. If $\kappa_1 < \frac{p'(k)-p''(0)}{k-0}$ and $k \geqslant q_n + 2$, then by lemma 9.b we have

$$\varkappa \leqslant 2k + 1 - \tfrac{k-q_n}{\kappa_2}$$

Using the inequality $\kappa_2 \leqslant 1$ we obtain $\varkappa \leqslant k + 1 + q_n$. If this inequality holds for $k = q_n + 2$, i.e., if $\kappa_1 < \frac{p'(q_n+2)}{q_n+2} = \frac{1}{q_n+2}$, then we have $\varkappa \leqslant 2q_n + 3$. □

5 Low bounds for spatial derivatives of solutions of Navier-Stokes system

In [2] low bounds of the L_∞ time-space norm of solutions of the nD free Navier-Stokes system are obtained. In this section we amplify those estimates to the case of L_1 in time of the L_∞ in space norm.

The free Navier-Stokes system can be regarded as a special case of the equation (1.1) when $m = n$, $\boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{u}$ and $\boldsymbol{h}$ represent the gradient of the pressure term. This observation allows us to extend the results of previous sections to solutions of the Navier-Stokes system.

In this section we consider the dynamics of a vector field $\boldsymbol{u} = \boldsymbol{u}(t,\boldsymbol{x})$ on the torus $\mathbb{T}^n = {}^{\mathbb{R}^n}\!/_{\ell\mathbb{Z}^n}$ described by the Navier-Stokes system:

$$\partial_t \boldsymbol{u} + \nabla_{\boldsymbol{u}}\boldsymbol{u} = \nu\Delta\boldsymbol{u} + \nabla p(t,\boldsymbol{x}), \tag{5.1}$$

$$\operatorname{div}\boldsymbol{u} = 0, \tag{5.2}$$

with a positive viscosity ν. We assume that the initial state $\boldsymbol{u}(0,\boldsymbol{x}) = \boldsymbol{u}_0$ is C^2-smooth. Let $T_0 = T_0(\boldsymbol{u}_0)$ be supremum of existence time-interval, i.e., the smooth solution of the system (5.1),(5.2) exists for $t \in [0, T_0)$. There are lower estimates for T_0; for example, in the paper [11] it is shown that $T_0 \geqslant \frac{C(n)\,\ell^2}{\nu R^2(1+\max\{0,\log R\})^2}$, where $R = \frac{\ell}{\nu}\,|\boldsymbol{u}_0|_{L_\infty}$ is the Reynolds number. It is clear that if $T_0 < \infty$, then $|\boldsymbol{u}(t,\cdot)|_{L_\infty} \to \infty$ as $t \to T_0$.

Remark 1. Since the mean value of $\boldsymbol{u}$ is constant, then the spatial derivatives of $\boldsymbol{u}$ blow up with $|\boldsymbol{u}(t,\cdot)|_{L_\infty}$. Due to this reason we assume that $\max\limits_{j=1,\dots,n}\left|\frac{\partial^k \boldsymbol{u}(t,\cdot)}{\partial x_j^k}\right|_{L_\infty} = \infty$ for $t \geqslant T_0$ and any $k \geqslant 1$.

Theorem 8. *Suppose that the initial state $\boldsymbol{u}_0$ is a non-degenerate vector field. Then there exist a ν-independent positive real numbers T and $\varkappa_2$, $\varkappa_3$, $\varkappa_4$,... such that for any $\nu > 0$ and for any $k \geqslant 2$ we have:*

$$\max\Big\{\max_{j=1\dots n} \frac{1}{T}\int_0^T \sup_{\boldsymbol{x}\in\mathbb{T}^n} \Big|\tfrac{\partial^k \boldsymbol{u}}{\partial {x_j}^k}(t,\boldsymbol{x})\Big|\,dt\,,\ \max_{j=1\dots n} \frac{1}{T}\int_0^T \tfrac{1}{\nu^{k/2}} \sup_{\boldsymbol{x}\in\mathbb{T}^n} \big|\tfrac{\partial^{k-1} p}{\partial x_j^{k-1}}(t,\boldsymbol{x})\big|dt\Big\} \geqslant \frac{\varkappa_k}{\nu^{k/2}}\,. \tag{5.3}$$

As the next result shows, the assumption of non-degeneracy cannot be removed. To show this we can restrict ourself to the 2D case since any solution of the 2D Navier-Stokes system generates a solution of a higher-dimensional system by adding zeroth components of the field $\boldsymbol{u}$ and fictitious variables. The non-degeneracy is preserved by this operation. We remark that in the 2D case the non-degeneracy is a necessary and sufficient condition for theorem 8.

Lemma 11. *Let $\boldsymbol{u}$ be a solution of 2D Navier-Stokes system with a degenerate initial state. Then for any $t > 0$ and $k \geqslant 0$ we have* $\max|\boldsymbol{u}(t,\boldsymbol{x})|_{C^k(\mathbb{T}^2)} \leqslant \max|\boldsymbol{u}(0,\boldsymbol{x})|_{C^k(\mathbb{T}^2)}$ *and* $\nabla p \equiv 0$.

Proof. In the 2D case any degenerate initial state has the form:

$$\boldsymbol{u}_0(\boldsymbol{x}) = \tbinom{b_2}{-b_1}\varphi_0(b_1x_1 + b_2x_2) + \tbinom{c_1}{c_2} \tag{5.4}$$

for a suitable function $\varphi_0(\cdot)$ and real numbers b_1, b_2, c_1, and c_2 (see theorem 3). In this case, the solution of the Cauchy problem for (5.1), (5.2) remains of the form (5.4):

$$\boldsymbol{u}(t,\boldsymbol{x}) = \tbinom{b_2}{-b_1}\varphi(t, b_1x_1 + b_2x_2) + \tbinom{c_1}{c_2}\,,$$

where the function φ satisfies the following linear parabolic equation with constant coefficients:

$$\varphi_t + (b_1c_1 - b_2c_2)\varphi' = (b_1^2 + b_2^2)\nu\varphi''\,.$$

Therefore derivatives of $\boldsymbol{u}$ are decreasing and $\nabla p \equiv 0$. □

We note that in the 2D case the class of periodic solutions with degenerate initial state coincides with the class of periodic solutions with $\nabla p \equiv 0$. (For the Euler equation this is true for any dimension, see theorem 4) Indeed, if the initial state is degenerate, then $\nabla p \equiv 0$ due to lemma 11. To show the converse we apply the identity

$$\Delta p(t,x) = -2I_2(t,x), \tag{5.5}$$

where $I_k(t,x)$ are the coefficients of the characteristic polynomial of the matrix $\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}(t,\boldsymbol{x})$:

$$\det\big(\tfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} - \lambda\mathbf{1}\big) = (-\lambda)^n + (-\lambda)^{n-1}I_1(t,x) + \dots + I_n(t,x).$$

The degeneracy of the initial state is equivalent to the condition $I_1(0,\cdot) \equiv I_2(0,\cdot) \equiv \ldots \equiv I_n(0,\cdot) \equiv 0$. The coefficient $I_1 \equiv 0$ due to (5.2).

The identity (5.5) is also valid in higher dimensions. For the proof one can take the divergence of (5.1), (5.2) and note that $\operatorname{div} \nabla_{\boldsymbol{u}}\boldsymbol{u} = -2I_2(t,x)$.

Proof of the theorem. As we will show, the assertion of the theorem holds for any $\boldsymbol{u}$ and p that satisfy (5.1) (we note that equation (5.2) is not used in our calculations). First, from theorem 2 we find $T = T(\boldsymbol{u}_0)$. If $T_0 < T$, then the left hand side of (5.3) is equal to infinity due to remark 1. Suppose $T < T_0$. Then from theorem 2 again, we find $c = c(\boldsymbol{u}_0)$ and $r_k = r_k(\boldsymbol{u}_0)$ ($k \geqslant 2$) such that if

$$\frac{1}{T}\int_0^T \sup_{\boldsymbol{x}\in\mathbb{T}^n} \big|\boldsymbol{\nabla} p(t,\boldsymbol{x})\big|dt < \tfrac{c}{2},$$

then for any $k \geqslant 2$ we have

$$\max_{j=1\dots n} \frac{1}{T}\int_0^T \sup_{\boldsymbol{x}\in\mathbb{T}^n} \Big|\tfrac{\partial^k \boldsymbol{u}}{\partial {x_j}^k}(t,\boldsymbol{x})\Big|\,dt \geqslant \tfrac{r_k}{\nu^{k/2}}\,.$$

If

$$\frac{1}{T}\int_0^T \sup_{\boldsymbol{x}\in\mathbb{T}^n} \big|\boldsymbol{\nabla} p(t,\boldsymbol{x})\big| dt \geqslant \tfrac{c}{2},$$

then for any $k \geqslant 2$ we have:

$$\max_{j=1\dots n} \frac{1}{T}\int_0^T \sup_{\boldsymbol{x}\in\mathbb{T}^n} \big|\tfrac{\partial^{k-1}p}{\partial x_j^{k-1}}(t,\boldsymbol{x})\big| dt \geqslant \tfrac{c}{2\ell^{k-1}\sqrt{n}}.$$

Inequality (5.3) is proved with the constant $\varkappa_k = \min\{r_k, \frac{c}{2\ell^{k-1}\sqrt{n}}\}$. □

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