On generalised equations of Burgers type with small viscosity

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Consider the one dimensional non-linear parabolic equation of Burgers type

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}f(u) = \nu u_{xx}. \tag{1}$$

Here ν a is positive real parameter (viscosity).

Theorem 1. *Suppose that $f \in \mathrm{Lip}(\mathbb{R})$[1] and the initial state $u_0 \in L_\infty(\mathbb{R})$; then the solution to the Cauchy problem for equation (1) satisfies*

$$\left|\frac{\partial u}{\partial x}(t,\cdot)\right|_{C^0(\mathbb{R})} \leqslant 5|u_0|_{L_\infty} \max\left(\frac{Lip(f)}{\nu}, \frac{1}{\sqrt{t\nu}}\right). \tag{2}$$

As a corollary for this Theorem we have:

$$\text{if } t \geqslant \frac{\nu}{Lip(f)^2} \quad \text{then} \quad \left|\frac{\partial u}{\partial x}(t,\cdot)\right|_{C^0(\mathbb{R})} \leqslant 5|u_0|_{L_\infty}\frac{Lip(f)}{\nu}.$$

Theorem 2. *Given $f \in C^k(\mathbb{R})$, $(k \geqslant 1)$ and an initial state $u_0 \in L_\infty(\mathbb{R})$; then there exist ν-independent constants C_k and $\tilde{c}_k$ such that for any $\nu > 0$ and for any $t \geqslant \tilde{c}_k\nu$ we have*

$$\left|\frac{\partial^k u}{\partial x^k}(t,\cdot)\right|_{C^0(\mathbb{R})} \leqslant C_k \frac{1}{\nu^k}. \tag{3}$$

Now we turn our attention to the case of small viscosity, i.e., $\nu \ll 1$. It turns out that if the viscosity ν is small then its power in (3) is optimal (in sense of Theorem 3 bellow).

Consider the so-called integrable case of equation (1), i.e., $f(u) = au^2$ with $a \neq 0$. In this case equation (1) can be reduced to the heat equation by so-called Hopf–Cole transformation ($u = -\frac{\nu}{a}\frac{\partial}{\partial x}\ln\varphi$). We remark that this transformation was known by V. Florin (see [1]) before Hopf and Cole. Using the explicit formula for the solution one can get

Theorem 3. *Suppose $f(u) = au^2$ with $a \neq 0$. Given $u_0 \in C_b^1(\mathbb{R})$ such that the Cauchy problem*

$$\begin{cases} u_t + 2auu_x = 0 \\ u(0,x) = u_0(x) \end{cases}$$

develops shocks for $t > T$. Then for any $t > T$ we have

$$\limsup_{\nu \to 0} \left|\nu^k \frac{\partial^k u}{\partial x^k}(t,\cdot)\right|_{C^0(\mathbb{R})} > 0. \tag{4}$$

REFERENCES

[1] V.A.Florin, *Some of the simplest nonlinear problems arising in the consolidation of wet soil.* Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1948**, No. 9(1948), 1389–1402.

[1] we actually need the condition $f \in \mathrm{Lip}(\operatorname{ess\,min} u_0, \operatorname{ess\,max} u_0)$.