Research Statement

Andrei Biryuk

My research is in applied mathematics, focusing mainly on non-linear PDEs – analytic and numerical methods. I have worked extensively with the Navier-Stokes equation, Burgers' equation, Boltzmann equation and others. I am also interested in the probabilistic approaches to hydrodynamics-type PDEs, ergodic theory, dynamical systems and statistical hydrodynamics.

1. Navier-Stokes:

It is a problem of greatest interest to prove that there exists a weak solution of the Navier-Stokes system with no singular space-time points[1]. Indeed, such solution is automatically strong and Serrin weak-strong uniqueness theorem would imply the uniqueness of the weak solution for the Navier-Stokes system. Moreover, the Serrin regularity theorem would imply that this solution is regular.

The famous Scheffer and Caffarelli-Kohn-Nirenberg theorem states that *if* a weak solution of the Navier-Stokes system is *suitable*, i.e., satisfies the *local* in time-space energy dissipation condition, then the set of singular points can not be of too large measure. Precisely, this means that the (parabolic) Hausdorff 1D measure of this set must be zero.

Due to the lack of a uniqueness theorem, every method of construction of a weak solution gives (generally speaking) its own type of a weak solution. It is then curious to know which of them are suitable. It is surprising that while for some methods (e.g. Leray or super-viscosity approximations) the suitability can be proven quite easily, for the famous Galerkin procedure of Hopf this is still an unknown fact.

Recently we've discovered an argument, which shows that if two weak solutions of NS have disjoint time-space singular sets, then both singular sets are empty, solutions coincide and are smooth [4].

2. Burgers/Schrodinger: The subject of investigation is the generalized multidimensional (not necessarily potential initial vector field) Burgers equation with small viscosity (the turbulent regime) and the propagation of shock waves. My PhD thesis, supervised by Prof. Kuksin, was on estimates for spatial derivatives of solutions for quasilinear Burgers equation with small viscosity. While working on this topic, I discovered several facts that I consider to be deep and that, surprisingly, were not known. One of them is that the Burgers equation does not have the Tikhonov non-uniqueness effect, unlike the heat equation [10].

I plan to study the spectral properties of solutions with very small viscosity and the corresponding motion of the continuum. Progress in this direction will lead to a better understanding of the phenomenon of turbulence. A possible approach to attacking this problem is to study the viscosity solutions of the fluid dynamics type equations. I am also considering the non-classical case, in particular, the equations with a generalized convective term represented by an arbitrary flow-type non-linearity. In the most non-trivial case, the numbers of dependent and independent variables are both greater than one, and are not necessarily equal. I have already obtained a number of interesting and promising relations for the solutions of such equations [9]. They are formulated in terms of the geometry of some additional objects, related to the solutions. For example, one can derive a lot of information just from the fact that a small ball around a moving particle is not going to collapse.

I have found that for certain non-degeneracy conditions on the initial data, lower bounds for spatial derivatives of the solutions of hydrodynamics-type equations can be established [9]. This is also applicable to the Navier-Stokes system [12]. Moreover, for the Navier-Stokes system, the converse is also true, i.e. no such bounds exist in the degenerate case [12]. I hope to incorporate these facts, as well as the future results in this area, into a general "theory of turbulence". The next step would be to study hydrodynamics on manifolds with non-trivial geometry in the turbulent regime. The same technique, (with modifications) applies to the Schrodinger equation. I plan to pursue this direction in the nearest future. Paper [1] initiates a series of works in this direction.

3. Boltzmann: The Boltzmann equation $f_t + v \cdot \nabla_x f = Q(f)$, where Q is the Boltzmann nonlinear (integral) collision operator, describes the evolution of the mass probability density function $f(x, v, t)$ for particles distribution in the position-velocity phase space. A new averaging property of the collision operator gives us a strong Lipschitz property for Q. It is a remarkable fact that we consider the space-inhomogeneous case. Then, the entropy

[1] A point (t, x) is singular if the solution is unbounded in any time-space neighborhood of this point.

dissipation property implies a new ordinary integro-differential inequality (OIDI) for the L^∞-norm. The conditions when this OIDI is globally solvable can be described explicitly, this is the first part of our results. For the second part, we've noticed that, with some restrictions, we can use the Bony/Cercignani functional instead of the entropy, to control the L^∞ norm. The above results are described in [3]. In [5] we describe a higher dimensional version, as well as give more refined properties of solutions. With our ideas, we elaborate on the ideas, originally due to Grad, DiPerna, Lions and others.

4. Other: I have recently started working on several projects involving the probabilistic approaches to hydrodynamics-type PDEs, ergodic theory, dynamical systems and statistical hydrodynamics [2,8,11]. I believe that many intriguing open problems exist at the turn of these research areas. Following Kolmogorov, studying the phenomenon of turbulence is equivalent to analyzing the statistical properties of the relevant invariant measures. I have already shown, that the class of microcanonical measures for the 2D Euler equation is different from the class of measures that are limits of the stationary distributions for randomly forced 2D NS equations, as the viscosity tends to zero [2]. This is due to the fact that these two classes have different statistical properties. Other problems I would like to explore include the Hamiltonian formalism in PDE, KAM theory and resonances.

List of Publications

[1] Biryuk A. *Lower bounds for derivatives of solutions for NLS.* Submitted to "Communications in Partial Differential Equations".
[2] Biryuk A. *On Invariant Measures of the 2D Euler Equation.* J. Stat. Physics. Vol. 122, no 4, (2006), pp. 597-616.
[3] Biryuk A., Craig W., Panferov V. *Strong solutions of the Boltzmann equation in one spatial dimension.* Comptes Rendus Mathematique Vol. 342, no. 11 (2006), pp. 843-848.
[4] Biryuk A., Craig W., Ibrahim S. *Construction of suitable weak solutions of the Navier-Stokes equation.* Contemporary Mathematics – AMS, 2006 (to appear)
[5] Biryuk A., Craig W., Panferov V. *Smoothness of solutions for the Boltzmann equation.* (in preparation)
[6] Biryuk A., Craig W., Ibrahim S. *Questions related to the uniqueness problem for weak solutions of the Navier-Stokes system.* (in preparation)
[7] Biryuk A., Kuksin S. *Remarks on the Navier-Stokes Equation.* (in preparation).
[8] Biryuk A. *About marginal density distribution.* Preprint 2005 (Russian. English summary).
[9] Biryuk A. *On Multidimensional Burgers Type Equations with Small Viscosity.*, Contributions to Current Challenges in Mathematical Fluid Mechanics. Series: Advances in Mathematical Fluid Mechanics, Galdi, Giovanni P.; Heywood, John G.; Rannacher, Rolf (Eds.) 2004, pp. 1-30. Birkhauser, Berlin.
[10] Biryuk A. *Note on the transformation that reduces the Burgers equation to the heat equation.* Preprint: mp_arc 03-370. (2003)
[11] Biryuk A. *Distribution of the component of the uniform distribution in a polyhedron.* Preprint 2003 (in Russian).
[12] Biryuk A. *On Spatial Derivatives of Solutions of the Navier-Stokes Equation with Small Viscosity.* "Uspekhi Mat. Nauk", Vol.57 No 1., (2002). English translation in "Russian Math. Surveys" **57** (2002), no. 1.
[13] Biryuk A. *Spectral Properties of Solutions of Burgers Equation with Small Dissipation.* Functional Analysis and Its Applications. Vol. 35., No 1., (2001), pp. 1-15.
[14] Biryuk A. *On generalized equations of Burgers type with small viscosity.* Proc. Int. Conf. "Differential Equations and Related Topics", Moscow–2001.
[15] Biryuk A. *Spectral Properties of Solutions of the Generalized Burgers Equation.* "BAMC-2000", Manchester–2000.
[16] Biryuk A. *Estimates for derivatives of the Burgers equations in terms of viscosity.*, Abstr. Tagungsbericht "Analytical and Statistical Approaches to Fluid Models", Oberwolfach–2000.

The manuscripts are available at `http://www.math.mcmaster.ca/~abiryuk`