Spatial derivatives of solutions of the Navier–Stokes equation with low viscosity

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We shall consider the dynamics of a vector field $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ on the n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\ell\mathbb{Z}^n$ described by the Navier–Stokes system

$$\partial_t \mathbf{u} + \nabla_{\mathbf{u}} \mathbf{u} = \nu \Delta \mathbf{u} + \nabla p(t, \mathbf{x}), \tag{1}$$

$$\operatorname{div} \mathbf{u} = 0, \tag{2}$$

with positive viscosity ν. The initial data $\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0$ are assumed to be C^2-smooth. It is known that the corresponding classical solution of the system (1), (2) exists for $0 \le t < T_0 \le \infty$, and $|\mathbf{u}(t, \cdot)|_{L_\infty} \to \infty$ as $t \to T_0$ if $T_0 < \infty$. In [1] it was demonstrated that $T_0 \ge \dfrac{C(n)\ell^2}{\nu R^2(1 + \max\{0, \log R\})^2}$, where $R = \dfrac{\ell}{\nu}|\mathbf{u}_0|_{L_\infty}$ is the Reynolds number.

Definition. The vector field $\mathbf{u}_0$ is singular if its Jacobi matrix is nilpotent everywhere, that is, $(\partial \mathbf{u}_0(\mathbf{x})/\partial \mathbf{x})^n \equiv 0$. Otherwise it is said to be non-singular.

We consider the characteristic polynomial of the Jacobi matrix $\partial \mathbf{u}(t, \mathbf{x})/\partial \mathbf{x}$,

$$\chi_{t,\mathbf{x}}(\lambda) = \det\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} - \lambda \mathbf{1}\right) = (-\lambda)^n + (-\lambda)^{n-1} I_1(t, x) + \dots + I_n(t, x).$$

The singularity of the initial data is equivalent to the condition $I_1(0, \cdot) \equiv I_2(0, \cdot) \equiv \dots \equiv I_n(0, \cdot) \equiv 0$.

Theorem. *Suppose that the initial condition* $\mathbf{u}_0$ *is a non-singular vector field. Then there exist a positive number* T *and positive numbers* $\varkappa_2, \varkappa_3, \varkappa_4, \dots$ *independent of* ν *such that for any* $\nu > 0$ *and* $k \ge 2$

$$\sup_{[0,T]\times\mathbb{T}^n} \max_{j=1,\dots,n} \left\{ \max\left\{ \left|\frac{\partial^k u_1}{\partial x_j^k}\right|, \dots, \left|\frac{\partial^k u_n}{\partial x_j^k}\right|, \frac{1}{\nu^{k/2}} \left|\frac{\partial^{k-1} p}{\partial x_j^{k-1}}\right| \right\} \right\} \ge \frac{\varkappa_k}{\nu^{k/2}}. \tag{3}$$

As demonstrated by our result below, the non-singularity assumption cannot be relaxed. It suffices to consider the two-dimensional case, since any solution of the two-dimensional Navier–Stokes system gives rise to a solution of the Navier–Stokes system in higher dimensions obtained by including additional zero components of $\mathbf{u}$ and dummy variables, which preserves the singularity in the initial data. But in the two-dimensional case non-singularity is a necessary and sufficient condition for the assertion of the theorem to hold.

Proposition. *Suppose that* $n = 2$ *and the initial data are singular. Then* $\max|\mathbf{u}(t, \mathbf{x})|_{C^k(\mathbb{T}^2)} \le \max|\mathbf{u}(0, \mathbf{x})|_{C^k(\mathbb{T}^2)}$ *and* $\nabla p \equiv 0$ *for any* $t > 0$ *and* $k \ge 0$.

Proof. In the two-dimensional case singular vector fields can be classified. A vector field being singular in this case is equivalent to its being divergence-free and having the determinant of the Jacobi matrix identically equal to zero. The first condition implies that there is a function $\psi\colon \mathbb{R}^2 \to \mathbb{R}$ such that $\operatorname{curl}\psi = \mathbf{u}_0(\mathbf{x})$, where $\operatorname{curl}\psi = \begin{pmatrix} \partial\psi/\partial x_2 \\ -\partial\psi/\partial x_1 \end{pmatrix}$. Now the second condition holds if and only if the graph of ψ in $\mathbb{R}^3$ is a surface with Gaussian curvature identically equal to zero. We find that the assumptions of the cylinder theorem hold: *A complete two-dimensional surface in* $\mathbb{R}^3$ *with Gaussian curvature identically equal to zero must be a generalised cylinder, that is, the Cartesian product of a straight line and a curve in an orthogonal plane.* This theorem was first obtained by Pogorelov ([2], Theorem 6) and slightly later by Hartman and Nirenberg.

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Applying this theorem, we find that every two-dimensional periodic singular vector field has the form

$$\mathbf{u}_0(\mathbf{x}) = \begin{pmatrix} b_2 \\ -b_1 \end{pmatrix} \varphi_0(b_1x_1 + b_2x_2) + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

for some function $\varphi_0(\,\cdot\,)$ and real numbers b_1, b_2, c_1, and c_2. For details see [3], Theorem 3. In this case the solution of the Cauchy problem for equation (1) preserves its form:

$$\mathbf{u}(t,\mathbf{x}) = \begin{pmatrix} b_2 \\ -b_1 \end{pmatrix} \varphi(t, b_1x_1 + b_2x_2) + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

where φ satisfies the linear parabolic equation $\varphi_t + (b_1c_1 - b_2c_2)\varphi' = (b_1^2 + b_2^2)\nu\varphi''$ with constant coefficients. It follows that the derivatives are decreasing and $\nabla p \equiv 0$. The proposition has been proved.

We observe that in the two-dimensional case solutions with singular initial data describe the class of solutions of the Navier–Stokes system for which $\nabla p \equiv 0$. In one direction this follows from the proposition just proved. In the opposite direction it follows from the identity $\Delta p(t,x) = -2I_2(t,x)$, which also holds in higher dimensions.

Proof of theorem. As we shall show, the assertion of the theorem holds for any functions $\mathbf{u}$ and p satisfying only (1) without the divergence-free condition (2). The proof is based on the fact that if $\mathbf{u}(t,\mathbf{x})$ is an arbitrary C^1-smooth vector field periodic in $\mathbf{x}$ such that the initial condition $\mathbf{u}_0 = \mathbf{u}(0,\mathbf{x})$ is a non-singular vector field, then there exist $T = T(\mathbf{u}_0)$ and $c = c(\mathbf{u}_0)$ such that

$$|\mathbf{u}_t + \nabla_{\mathbf{u}}\mathbf{u}|_{C^0([0,T]\times\mathbb{T}^n)} \geq c. \tag{4}$$

(See [3], Theorems 4, 5.) If $T_0 \leq T(\mathbf{u}_0)$, then we obtain (3) from the fact that $|\mathbf{u}(t,\,\cdot\,)|_{L_\infty} \to \infty$ as $t \to T_0$ while the mean value remains constant.

If $T_0 > T(\mathbf{u}_0)$, then (1) and (4) give

$$|\nu\Delta\mathbf{u} + \nabla p|_{C^0([0,T]\times\mathbb{T}^n)} \geq c. \tag{5}$$

If $|\nabla p|_{C^0([0,T]\times\mathbb{T}^n)} \geq \dfrac{c}{2}$, that is, $\left|\dfrac{\partial p}{\partial x_j}\right|_{C^0} \geq \dfrac{c}{2}$ for some j, then $\left|\left(\dfrac{\partial}{\partial x_j}\right)^{k-1} p\right|_{C^0} \geq \dfrac{c}{2\ell^{k-1}}$ for any $k \geq 2$, which proves (3) in this case.

Suppose that $|\nabla p|_{C^0([0,T]\times\mathbb{T}^n)} < c/2$. In this case we can regard (1) as the Burgers equation for $\mathbf{u}$ with bounded external force ∇p. Applying the maximum principle to this equation, we obtain $\sup_{[0,T]\times\mathbb{T}^n} |\mathbf{u}(t,\mathbf{x})| \leq |\mathbf{u}_0|_{L_\infty} + cT/2$, since (5) implies that $|\Delta\mathbf{u}|_{C^0([0,T]\times\mathbb{T}^n)} > c/(2\nu)$. Extrapolating the last two inequalities, we obtain for every $k \geq 2$ the inequality

$$\max_{j=1,\dots,n} \left|\left(\frac{\partial}{\partial x_j}\right)^k \mathbf{u}\right|_{C^0([0,T]\times\mathbb{T}^n)} \geq \frac{r_k}{\nu^{k/2}}$$

with a ν-independent positive constant r_k.

The inequality (3) is proved with $\varkappa_k = \min\{r_k, c/(2\ell^{k-1})\}$. This completes the proof of the theorem.

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Bibliography

[1] I. Kukavica and Z. Grujić, *J. Funct. Anal.* **152** (1998), 447–466.

[2] A. V. Pogorelov, *Dokl. Akad. Nauk SSSR* **111** (1956), 945–947. (Russian)

[3] A. Biryuk, "On multidimensional Burgers type equations with small viscosity", Submitted to *J. Math. Pures Appl.*; http://www.mccme.ru/~biryuk.

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